CONFORMAL INVARIANT FINITE QUANTUM ELECTRODYNAMICS

By
RADHEY SHYAM

PAKD.
PHY
1977
D
SHY
CON
TD
PHY/1977/D
Sh97c

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
JANUARY, 1977

CONFORMAL INVARIANT FINITE QUANTUM ELECTRODYNAMICS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
RADHEY SHYAM

to the

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JANUARY, 1977

CEN RY
Acc. 53972
2 MAY 1978

POST GRADUATE OFFICE
Submitted on 25/1/77

CERTIFICATE

Certified that the work presented in this thesis entitled, 'Conformal Invariant Finite Quantum Electrodynamics', by Radhey Shyam has been carried out under my supervision and that this has not been submitted elsewhere for a degree

Tulsi Dass

(Tulsi Dass)
Assistant Professor
Department of Physics
Indian Institute of Technology,
Kanpur-208016, INDIA

January, 1977

POST GRADUATE OFFICE
This thesis has been approved
for the award of the Degree of
Doctor of Philosophy (Ph.D.)
in accordance with the
regulations of the Indian
Institute of Technology Kanpur
Dated: 17/2/78

ACKNOWLEDGEMENTS

I am deeply indebted to Professor Tulsi Dass for introducing me into the subject and suggesting the present investigation I am thankful to him for his constant help throughout the work

My thanks are due to Professor Gyan Mohan from whom I was initiated into research and for many discussions on quantum electrodynamics My thanks are due to Professor R Ramachandran and Professor H S Mani from whom I have learnt much of theoretical physics

I am very grateful to Drs K Kapoor, Raghuvir Singh, J Maharana, S Krishna, V M Raval, V K Agrawal, Pankaj Sharan, M Prakash and M/S M S Sriram, R S Saratchandra and Anurag Jaiman for helping me in various ways

I am grateful to the S B F , and the Physics Department for providing material help in the preparation of the thesis

I am very grateful to Dr R L Verma and Nanda Ballabh Tiwari for constant encouragement

My thanks are due to Mr J K Misra for excellent typing of the manuscript, and to Mr H K Panda and Mr Lallu Singh for cyclostyling it nicely

Financial assistance from C S I R during early stages of the work is gratefully acknowledged.

Radhey Shyam

CONTENTS

SYNOPSIS

Thesis Entitled, 'Conformal Invariant Finite Quantum Electrodynamics', submitted by RADHEY SHYAM in Partial Fulfilment of the Requirements of the Ph D Degree to the Department of Physics, Indian Institute of Technology, Kanpur, January 1977

We modify Dirac-Maxwell Lagrangian for electromagnetic interaction by introducing exponential coupling of a massless scalar field (which is essentially scalar gravity) in a conformally invariant manner Using non-polynomial Lagrangian techniques, we show that the usual ultraviolet divergences are absent from the theory, the inverse of the minor coupling constant provides the cut off The method can be extended to other field theories; however we have carried out this program in some detail for quantum electrodynamics only

The thesis is divided into nine sections The first section is an introduction In Section II, we present the formalism of conformal invariant quantum electrodynamics in a self contained manner We set up a conformal invariant Lagrangian by using conformal covariant derivatives in the usual Poincare invariant Lagrangian and multiply the electron mass term by exp $(-f\sigma(x))$ where $\sigma(x)$ is the massless scalar field Then we apply a field

transformation to remove the exponential factor from the mass term, it then appears in the kinetic energy and interaction terms We use the modified electromagnetic interaction for our further calculations

In Section III we calculate the self energy of electron due to the modified electromagnetic interaction and identify the electron wave-function renormalization constant and the traditional finite part left after the renormalization The self mass and the renormalization constant do not have ultraviolet divergences to zeroth order in f, the traditional finite part is same as in conventional quantum electrodynamics

In Section IV, we present the calculation of photon self energy and identify photon wave function renormalization constant which is now finite and the traditional finite part of the polarization tensor left after the renormalization which agree with the conventional quantum electrodynamics (to zeroth order in f) The calculation with modified electromagnetic interaction alone is not gauge invariant The gauge invariance is restored by replacing the formal current operator by a strictly gauge invariant current operator (as suggested by Schwinger) and including the effects of modified kinetic energy terms

In Section V the vertex part has been considered The Ward identity upto zeroth order in f has been verified and the vertex renormalization constant and conventional finite part left after renormalization has been identified

In Section VI, we present a simple proof of finiteness of the theory to all orders

In Section VII, we calculate gravitational self energy of the electron due to interaction of the electron with scalar gravity only In lowest order the gravitational self energy of the electron is finite and is of the form $m^3f^2 \ln (mf)$

In Section VIII, we set up a conformal invariant Lagrangian for the interaction of charged and neutral pions with the electromagnetic field and scalar gravity and calculate $\pi^+-\pi^0$ mass difference The value obtained is somewhat higher than the experimental value

The last section is concerned with some concluding remarks

I INTRODUCTION

Ultraviolet infinities made their appearance in particle physics some seventy five years ago The first infinity arose in Lorentz's calculation of electron self-mass (δm) in classical electrodynamics He obtained $\delta m = e^2/R$ which is linearly divergent as $R \to 0$ for a point electron The divergence difficulty persisted even in quantum electrodynamics The first calculation of δm by Waller[1], using Dirac's single particle electron equation produced a quadratically divergent expression Using positron theory Weisskopf[2] was able to show that the severity of divergence could be reduced to logarithmic Nevertheless, the divergence problem persisted He obtained

$$\frac{\delta m}{m} = \frac{6\alpha}{4\pi} \lim_{R\to 0} \log \left(\frac{1}{Rm}\right) + \text{finite term}$$

Weisskopf also proved the important result that in the n-th order, $\frac{\delta m}{m}$ may diverge as fast as $\alpha^n \log \left(\frac{1}{Rm}\right)$ In view of this result, he ascribed a 'critical length' to positron theory

$$R_{critical} \simeq \frac{1}{m} \exp(-1/\alpha) \simeq 10^{-56}/m$$

This would be the length where the theory may be expected to need fundamental modification, whether realistic or mathematical Using Feynman's techniques Dyson[3] confirmed

Weisskopf's result and in addition showed that there is one more infinity, viz self charge at atleast in perturbation solution, in addition to the self mass δm Renormalization program circumvented the infinities but it was not a final cure One problem that theoreticians could not understand was, whether infinities are due to the perturbation expansion around the point e = 0, or they are intrinsic to the theory Recent work of Jaffe and Glimm[4] shows that logarithmic ultraviolet infinities persist even in non-perturbative solutions of certain field theories in two and three space dimensions with polynomial interactions From this it appears that infinities are intrinsic to the theory and therefore one should try to modify Dirac Lagrangian for electrodynamics

The natural question then is, what kind of modification one should have? The work on non-polynomial lagrangians[5] which have remarkable convergence properties suggests that one should have non-polynomial modification of the interaction An ad-hoc modification by nonpolynomial interaction has been considered by Budini and Calicci[6] Obviously, one would like to have a natural modification based on some general principle Now, it has been conjectured[7] that the nonpolynomial coupling

of gravitational field with matter may provide a natural damping of ultraviolet infinities. The conjecture was recently revived by Delbourgo, Salam and Strathdee[8]. In a subsequent paper Isham, Salam and Strathdee[9] show that gravity modified quantum electrodynamics has a natural ultraviolet cut off provided by the gravitational coupling constant.

However, nonpolynomial theories have their own problems. In particular, they suffer from Borel ambiguities and distribution theoretic ambiguities[10]. Lehmann and Pohlmeyer[11] have shown that the distribution theoretic ambiguities may be removed for localizable Lagrangians by employing physical criteria on large momentum behaviour of the superpropagators. Furthermore these localizable Lagrangians are free of Borel ambiguities and possess the desirable features of good field theories[10,11]. Therefore, one should work with localizable Lagrangians as done by Isham, Salam and Strathdee[12], by adopting exponential parametrization of the vierbein field. Although in principle it is possible to remove ultraviolet divergences by coupling gravity with matter, the practical calculations in this scheme are very complicated. Also, one cannot claim to have obtained a finite theory as one does not know how to take care of additional divergences introduced by self-couplings of gravity.

In this work[13], we propose a modification of quantum electrodynamics which circumvents the above difficulties of tensor gravity. The modification consists in introducing a scalar field having conformal invariant interaction with the electromagnetic and the matter field, the conformal symmetry being realized nonlinearly through an exponential interaction. This scalar field essentially corresponds to scalar gravity[14].

The conformal symmetry is known since 1910 when Cunningham[15] and Bateman[16] observed that Maxwell's equations are covariant not only under Lorentz group but also under the larger group i.e., the conformal group. In 1936 Page[17] developed a new special relativity recognized to be based on conformal group. This implied that not only all coordinate systems with constant relative velocity are equivalent, but in fact all those with constant relative acceleration are also equivalent. In the same year, Dirac[18] showed the equivalence of conformal transformations to pseudo-orthogonal transformation in six dimensional space. Taking special conformal transformations as coordinate transformations to uniformly accelerated frame, Fulton, Rohrlich and Witten[19] have studied its physical consequence. Castell[20] has considered analysis of space-time structure in elementary particle physics. Kastrup[21] has studied the role of conformal

invariance in quantum mechanics and has connected it with indefinite metric in Hilbert space [illegible][22] has studied conformal invariance in quantum field theory Many authors[23] have constructed the representations of the conformal group Also Mack and Salam[24] have constructed finite component field representations

Salam and Strathdee[25,26] considered the possibility of spontaneously broken conformal symmetry and thus constructed conformal invariant Lagrangians with non-linear realizations of conformal symmetry In their program, two fields, a vector field for covariant derivative and a scalar field for making the Lagrangian invariant, were introduced However Ellis[27] has pointed out that only the scalar field is sufficient

Here, in Section 2, we use the approach of Salam and Strathdee and Ellis to write the conformal invariant Lagrangian for quantum electrodynamics In Section 3, the calculation of the photon contribution to the self-mass of electron is presented and the wave-function renormalization constant obtained The calculation of traditionally finite part is also presented In Section 4, the calculation of self energy of the photon is presented A naive calculation leads to a non gauge invariant result But by a more careful definition of the electromagnetic current through a limiting procedure, and

the inclusion of the 'kinetic-energy modification term, the gauge invariance is restored to order f^0 where f is the minor coupling constant In Section 5, the vertex part has been treated In Section 6, the proof of finiteness of the theory to all orders is presented In Section 7, the contribution of scalar gravity to the self energy of electron has been calculated

As a practical application of the above considerations, we calculate $\pi^+-\pi^0$ mass difference in Section 8 We conclude in Section 9 with some remarks

II CONFORMAL INVARIANT QUANTUM ELECTRODYNAMICS

The conformal group of space time consists of the following transformations

(1) Inhomogeneous Lorentz transformations

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu} \qquad (2\ 1)$$

where, $$g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma} \qquad (2\ 2)$$

with $$g_{00} = +1 \quad \text{and} \quad g_{11} = -1 \qquad (2\ 3)$$

(2) Dilatations

$$x^{\mu} \to x'^{\mu} = e^{\lambda} x^{\mu}, \quad \lambda \text{ real} \qquad (2\ 4)$$

(3) Special conformal transformations

$$x^{\mu} \to x'^{\mu} = \frac{x^{\mu} + \beta^{\mu} x^2}{1+2\beta x + \beta^2 x^2} \qquad (2\ 5)$$

By a direct calculation, one can show that, for special conformal transformations,

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} g_{\mu\nu} = \frac{g_{\rho\sigma}}{(1+2\beta x + \beta^2 x^2)^2} \qquad (2\ 6)$$

Taking the determinant of both sides of (2 6), we get,

$$\left| \det \frac{\partial x'}{\partial x} \right| = \frac{1}{(1+2\beta x + \beta^2 x^2)^4} \qquad (2\ 7)$$

using this relation in equation (2 6) we get for special conformal transformations,

$$\frac{\partial y'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial y^{\sigma}} \; g_{\mu\nu} = \left| \det \frac{\partial x'}{\partial x} \right|^{1/2} g_{\rho\sigma} \qquad (2\ 8)$$

It is easy to see that the relation (2 8) is also true for dilatations and is clearly true for inhomogeneous Lorentz transformations Thus it is true for the entire conformal group

If we define the matrix $\Lambda^{\mu}_{\nu}(x)$ by,

$$\Lambda^{\mu}_{\nu}(x) = \left| \det \frac{\partial x'}{\partial x} \right|^{-1/4} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \qquad (2\ 9)$$

Λ^{μ}_{ν} is a Lorentz matrix obeying the relation (2 2). Under any general transformation, dx^{μ} transforms as

$$dx^{\mu} \rightarrow dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \qquad (2\ 10)$$

For conformal transformations, using relations (2 9) we get the transformation for coordinate differential as follows

$$dx^{\mu} \longrightarrow dx'^{\mu} = \left| \det \frac{\partial y'}{\partial x} \right|^{1/4} \Lambda^{\mu}_{\nu}(x)\, dx^{\nu} \qquad (2\ 11)$$

The line-element ds defined by

$$ds^{2} = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \qquad (2\ 12)$$

transforms under the conformal group

$$ds^2 \to ds'^2 = \det \left| \frac{\partial x'}{\partial x} \right|^{1/2} ds^2 \qquad (2.13)$$

The line element ds^2 is not invariant under conformal transformations, only $ds^2 = 0$ is left invariant. It is possible to define an invariant length $d\bar{s}^2$ as follows,

$$d\bar{s}^2 = g_{\mu\nu} \lambda(x) \, dx^\mu dx^\nu \qquad (2.14)$$

where $\lambda(x)$ is a function of x. If one requires the invariance of this, $\lambda(x)$ should transform as,

$$\lambda(x) \to \lambda'(x') = \left| \det \frac{\partial x'}{\partial x} \right|^{-1/2} \lambda(x) \qquad (2.15)$$

If we parametrize $\lambda(x)$ in terms of a field $\sigma(x)$ as

$$\lambda(x) = e^{-2f\sigma(x)} \qquad (2.16)$$

(where f is a coupling constant called minor coupling constant), we obtain,

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \qquad (2.17)$$

where $$\bar{g}_{\mu\nu} = g_{\mu\nu} e^{-2f\sigma(x)} \qquad (2.18)$$

using Eq. (2.15), we see that,

$$e^{f\sigma'(x')} = \left| \det \frac{\partial x'}{\partial x} \right|^{1/4} e^{f\sigma(x)} \qquad (2.19)$$

and $$\sigma'(x') = \sigma(x) + \frac{1}{4f} \log \left| \det \frac{\partial x'}{\partial x} \right| \qquad (2.20)$$

Now we construct the representations of the conformal group. A convenient method of constructing representation of the conformal group from those of the inhomogeneous Lorentz group is based on the fact that $\Lambda^{\mu}_{\nu}(x)$ defined by Eq. (2.9) is a Lorentz matrix. Then, given a set of fields $\psi_\alpha(x)$ which transforms under inhomogeneous Lorentz transformations as,

$$\psi'(x') = D(\Lambda)\psi(x),$$

then the transformation,

$$\psi'(x') = \left|\det\frac{\partial x'}{\partial x}\right|^{\ell_\psi/4} D(\Lambda(x))\ \psi(x) \qquad (2.21)$$

provides a representation of the conformal group, where ℓ_ψ is the scale dimension of the field ψ. Now,

$$\frac{\partial\psi'(x')}{\partial x'_\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\left|\det\frac{\partial x'}{\partial x}\right|^{\ell_\psi/4} D(\Lambda(x))$$

$$\times\left[\frac{\partial\psi(x)}{\partial x^\nu} + \frac{1}{4}\left(\ell_\psi g_{\nu\rho} - iS_{\nu\rho}\right)\right.$$

$$\left.\times\,\partial^\rho\left(\log\left|\det\frac{\partial x'}{\partial x}\right|\right)\psi\right] \qquad (2.22)$$

In Eq. (2.22) $S_{\mu\nu}$ are matrices representing Lorentz generators in the representation of the field ψ for spin 1/2 and spin 1 fields, $S_{\mu\nu}$ have the following form

$$\text{Spin } 1/2 \qquad S_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu] \tag{2 23a}$$

$$\text{Spin } 1 \qquad S^{\rho\sigma\mu\nu} = g^{\rho\mu} g^{\sigma\nu} - g^{\sigma\mu} g^{\rho\nu} \tag{2 23b}$$

Thus $\partial_\mu \psi(x)$ does not transform covariantly Now we can construct covariant derivatives using field $\sigma(x)$ introduced in the beginning The covariant derivative is given by,

$$\Delta_\mu \psi = \partial_\mu \psi - f(\ell_\psi \ g_{\mu\nu} - i \ S_{\mu\nu}) (\partial^\nu \sigma) \psi \tag{2 24}$$

Under conformal group covariant derivative $\Delta_\mu \psi$ transforms as

$$(\Delta_\mu \psi)'(x') = \left| \det \left(\frac{\partial x'}{\partial x}\right) \right|^{(\ell_\psi - 1)/4} \Lambda^\nu{}_\mu (x) \ D(\Lambda(x))$$

$$\times \Delta_\nu \psi(x) \tag{2 25}$$

Proof of Eqs (2 22) and (2 25) is presented in the Appendix A

To ensure the invariance of action one should have the Lagrangian density transforming a

$$L(x) \to L'(x') = \det \left| \frac{\partial x'}{\partial x} \right|^{-1} [L(x) + \text{ four divergence}]$$

$$\tag{2 26}$$

To make the Lagrangian conformal invariant, we replace ordinary derivatives by covariant derivatives and multiply the terms of Lagrangian by suitable powers of exp $(f\sigma(x))$ To this Lagrangian, we should add the kinetic energy term for the σ-field

The $e^{-f\sigma}$ - field transforms as

$$e^{-f\sigma'(x')} = \left|\det \frac{\partial x'}{\partial x}\right|^{-1/4} e^{-f\sigma(x)}$$

The covariant derivative of $e^{-f\sigma(x)}$ vanishes identically Therefore we employ for the σ-part of the Lagrangian

$$L_\sigma = \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma e^{-2f\sigma} \tag{2 27}$$

L_σ transforms as,

$$L'_\sigma (x') = \left|\det \frac{\partial x'}{\partial x}\right|^{-1} [L_\sigma - \frac{1}{2f} \partial^\mu$$

$$\times \left\{ \partial_\mu \left(\frac{1}{4f} \log \left|\det \frac{\partial x'}{\partial x}\right|\right) e^{-2f\sigma} \right\}] \tag{2 28}$$

and is seen to transform as a scalar density with scale dimension -4, upto a four divergence

We are now in a position to construct the conformal invariant Lagrangian for quantum electrodynamics One can easily seethat, for the photon field A_μ,

$$\Delta_\mu A_\nu - \Delta_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv F_{\mu\nu} \tag{2 29}$$

Next, we define a derivative of the Dirac field Ψ which is covariant under gauge transformations as well as under conformal transformation, i e

$$D_\mu \Psi = (\partial_\mu + ie A_\mu)\Psi + f (3/2 g_{\mu\nu} + iS_{\mu\nu}) \partial^\nu \sigma \Psi \tag{2 30}$$

Then, the conformal invariant Lagrangian for quantum electrodynamics of spin - 1/2 particles is

$$= \frac{1}{2} [\bar{\psi} \gamma^{\mu} D_{\mu} \psi - D_{\mu} \bar{\psi} \gamma^{\mu} \psi] - m \bar{\psi} \psi e^{-f\sigma}$$

$$- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial^{\mu} \sigma \partial_{\mu} \sigma e^{-2f\sigma} \qquad (2\ 31)$$

To remove the exponential factor from the mass term in Eq (2 31), we perform the transformation,

$$\psi (x) = \psi_1 e^{-\frac{1}{2} f\sigma} \qquad (2\ 32)$$

Then,

$$L = L_o + L_{em} + L' \qquad (2\ 33)$$

where,

$$L_o = \frac{1}{2} [\bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{\mu} \psi] - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$+ \frac{1}{2} \partial^{\mu} \sigma \partial_{\mu} \sigma \qquad (2\ 34a)$$

$$L_{em} = -e \bar{\psi} \gamma^{\mu} \psi A_{\mu} e^{f\sigma} \qquad (2\ 34b)$$

$$L' = L'_1 + L'_2$$

$$L'_1 = \frac{1}{2} [\bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{\mu} \psi] (e^{f\sigma} - 1) \qquad (2\ 34c)$$

and,

$$L'_2 = \frac{1}{2} \partial_{\mu} \sigma \partial_{\mu} \sigma (e^{-2f\sigma} - 1) \qquad (2\ 34d)$$

The normal ordering in Eqs (2 34) is understood

It is easy to see that the field equation satisfied by σ-field is

$$\Box\left(e^{-2f\sigma(x)}\right) = -2f^2\,\theta^{\mu}_{\mu} \tag{2 35}$$

where,
$$\theta^{\mu}_{\mu} = \frac{1}{2}\left[i\bar{\psi}\gamma^{\mu} D_{\mu}\psi - D_{\mu}\bar{\psi}\,\gamma^{\mu}\psi\right] e^{f\sigma(x)}$$

$$-\partial^{\mu}\sigma\,\partial_{\mu}\sigma\; e^{-2f\sigma(x)} \tag{2 36}$$

is the trace of the energy-momentum tensor Thus the field $\sigma(x)$ is essentially scalar gravity

Next, we relate the minor coupling constant f to the gravitational constant G

Under weak field approximation, we have,

$$e^{-2f\sigma(x)} = 1-2f\sigma(x) \tag{2 37}$$

Now, in non-relativistic approximation,

$$\theta_{ii} = \theta_{i0} \approx 0\;,\quad \theta^{0}_{0} = \rho \tag{2 38}$$

Now taking weak and static-field approximation for the left side of the equation (2 35) and non-relativistic approximation for the right side of equation, we get,

$$\nabla^2\sigma(x) = -f\rho \tag{2 39}$$

Now,
$$\bar{g}_{00} = g_{00}\,e^{-2f\sigma(x)} = 1-2f\sigma(x) = 1+2\phi \tag{2 40}$$

where $\emptyset$ is the gravitational potential [28] From equation (2 40),

$$\sigma(x) = -\frac{1}{f}\ \emptyset \tag{2 41}$$

Now $\emptyset$ satisfies the equation

$$\nabla^2\emptyset = 4\pi G\rho \tag{2 42}$$

Therefore, from (2 39), (2 41) and (2 42), we get,

$$f^2 = 4\pi G \tag{2 43}$$

The numerical value of f is,

$$f \approx \frac{1}{9} \times 10^{-18}\ (\mathrm{GeV})^{-1} \tag{2 44}$$

III ELECTRON SELF-ENERGY

In this section, we calculate the lowest order self-energy of electron using L_e. The Feynman diagram contributing to the self-energy is given in Fig. 1. The matrix

Electron line

Photon line

Superpropagator

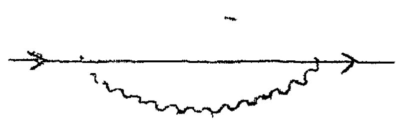


Fig. 1 Electron Self Energy

element for this process is given by

$$\Sigma(p) = ie^2 \int d^4x \; \gamma^\mu S_F(x) \gamma_\mu D_F(x) \; e^{f^2 D_F(x)} \; e^{ipx} \qquad (3.1)$$

where the convention for Dirac matrices is that of Bjorken-Drell and

$$S_F(x) = \frac{1}{(2\pi)^4} \int d^4q \; \frac{\gamma \cdot q+m}{q^2-m^2+i\epsilon} \; e^{-iq \cdot x}$$

$$D_F(x) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2+i\epsilon} \; e^{ik \cdot x}$$

$$= -\frac{1}{4x^2} \frac{1}{x^2-i\epsilon} \qquad (3.2)$$

Now,

$$D_{+}(x)\, e^{f^2 D_{F}(x)} = \sum_{n=0} \frac{(f^2)^n}{n!} [D_{F}(x)]^{n+1}$$

$$= \frac{1}{2\pi i} \int_{c} dz\, \Gamma(-z)\, (-f^2)^z\, [D_{F}(x)]^{z+1} \qquad (3\ 3)$$

where c is a contour enclosing positive real axis and running in the clockwise direction The contour c can be deformed to a contour c_1 parallel to the imaginary axis and $-1 < \mathrm{Re}\, z < 0$ on the contour c_1 The extra contribution coming from a circular arc of infinite radius can be seen to be zero Thus we get for the above expression,

$$D_{F}(x)e^{f^2 D_{F}(x)} = \int_{\alpha-i\infty}^{\alpha+i\infty} dz\, \Gamma(-z)\, (-f^2)^z\, [D_{F}(x)]^{z+1} \qquad (3\ 4)$$

where $-1<\alpha<0$

Now using Gelfand-Shilov formula,[30]

$$D_{F}^{z}(x) = -\frac{1}{(2\pi)^4} \int d^4k\, (4\pi)^{2-2z}\, \frac{\Gamma(2-z)}{\Gamma(z)}\, (-k^2)^{z-2}\, e^{ikx} \qquad (3\ 5)$$

and performing the x-integration and one momentum integration, we obtain,

$$\Sigma(p) = \frac{1}{2\pi i} \int dz\, \Gamma(-z)\, \Sigma(p, z) \qquad (3\ 6)$$

where, $\Sigma(p,z) = F(z) \int \frac{d^4k}{(2\pi)^4}\, \gamma^{\mu}\, \frac{\gamma(p-k)+m}{(p-k)^2-m^2}\, \gamma_{\mu}(-k^2)^{z-1} \qquad (3\ 7)$

and,

$$F(z) = ig^2 (-f^2)^z (4\pi)^{-2z} \frac{\Gamma(1-z)}{(\;-z)} \qquad (3\ 8)$$

Now, by Lorentz invariance

$$\Sigma(p, z) = \not{p}\, A(p^2,z) + m\, B(p^2,z) \qquad (3\ 9)$$

where,

$$A(p^2 z) = \frac{1}{4p^2} \mathrm{Tr}\, (\not{p}\Sigma(p,z)) \qquad (3\ 10)$$

and

$$B(p^2,z) = \frac{1}{4m} \mathrm{Tr}\, (\Sigma(p,z)) \qquad (3\ 11)$$

From Eqs (3 10) and (3 11) we get,

$$B(p^2,z) = 4F(z)\ I\ (p^2, m^2, z) \qquad (3\ 12)$$

where, $$I(p^2,m^2,z) = \int \frac{d^4k}{(2\pi)^4} \frac{(-k^2)^{z-1}}{(p-k)^2-m^2} \qquad (3\ 13)$$

and,

$$A(p^2,z) = \frac{F(z)}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{-2p^2 + 2pk}{(p-k^2)-m^2} (-k^2)^{z-1} \qquad (3\ 14)$$

In Eqs (3 13) and (3 14), we take p to be space-like For time like p, an analytic continuation can be performed The integral $I(p^2,m^2,z)$ has been calculated in Ref (12), however, as there is a small error in that calculation, we give a calculation of the integral below

Using α-parametric representation

$$\frac{1}{(p-k)^2-m^2} = - \int_0^\infty \exp\, [\alpha\{(p-k)^2-m^2\}] \qquad (3\ 15)$$

and,

$$(-k)^{z-1} = \frac{1}{\Gamma(1-z)} \int_0^\infty e^{\alpha k^2} \; {}^{-z} \qquad (3.16)$$

We get,

$$I(p^2, m^2, z) = -\frac{1}{\Gamma(1-z)} \int \frac{d^4k}{(2\pi)^4} \int_0^\infty d\alpha_1 \, d\alpha_2$$

$$\times \exp\left[\alpha_1 \{(p-k)^2 - m^2\} + \alpha_2 k^2\right] \alpha_2^{-z} \qquad (3.17)$$

where,

$$d^4k = d^3k \, dk_0 = i d^3k \, dk_4 = i(d^4k)_{Eucl}$$

$$k^2 = -\bar{k}^2 + k_0^2 = -|\vec{k}|^2 - k_4^2 = -k^2_{Eucl}$$

Shifting the origin of integration variables in (3.17),

$$I(p^2, m^2, z) = -\frac{1}{\Gamma(1-z)} \int \frac{d^4\ell}{(2\pi)^4} \int_0 d\alpha_1 \, d\alpha_2$$

$$\exp\left[(\alpha_1+\alpha_2)\ell^2\right] \exp\left[+\frac{\alpha_1 \alpha_2}{\alpha_1+\alpha_2} p^2 - \alpha_1 m^2\right] \alpha_2^{-z}$$

$$(3.18)$$

where,

$$\ell_\mu = k_\mu - \frac{\alpha_1}{\alpha_1+\alpha_2} p_\mu \qquad (3.19)$$

Integrating over ℓ, using the formula,

$$\int \frac{d^4\ell}{(2\pi)^4} e^{(\alpha_1+\alpha_2)\ell^2} = \frac{1}{16\pi^2(\alpha_1+\alpha_2)^2} \qquad (3.20)$$

We get,

$$I(p^2,m^2 z) = -\frac{1}{16\pi^2 \Gamma(1-z)} \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2}{(\alpha_1+\alpha_2)^2}$$

$$\times \exp\left[\frac{\alpha_1 \alpha_2}{\alpha_1+\alpha_2} p^2 - \alpha_2 m^2\right] \alpha_2^{-z} \qquad (3\ 21)$$

To evaluate (3 21), we introduce the change of variables, $\alpha_1 = rt$, $\alpha_2 = r(1-t)$ to obtain

$$I(p^2,m^2,z) = -\frac{1}{16\pi^2 \Gamma(1-z)} \int_0^\infty dr \int_0^1 dt\ r^{-z-1} (1-t)^{-z}$$

$$\exp\left[rt(1-t)p^2 - m^2 rt\right] \qquad (3\ 22)$$

Defining,

$$s = \left[m^2 t - t(1-t)\ p^2\right] r \qquad (3\ 23),$$

integrating over s by using the formula,

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1}\ e^{-t},$$

We get,

$$I(p^2,m^2,z) = -\frac{1(m^2-p^2)^z\ \Gamma(-z)}{16\pi^2 \Gamma(1-z)} \int_0^1 dt\ t^z (1-t)^{-z}$$

$$\times\left(1 + \frac{p^2}{m^2-p^2}\ t\ \right)^z \qquad (3\ 24)$$

$$= -\frac{1\ \Gamma(-z)}{16\pi^2 \Gamma(1-z)}\ \frac{\Gamma(1+z)\ \Gamma(1-z)}{\Gamma(z)} (m^2 - p^2)^z$$

$$\times F\left(-z,\ 1+z,\ 2,\ -\frac{p^2}{m^2-p^2}\ \right) \qquad (3\ 25)$$

where F is the hypergeometric function, defined by,

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\ t^{b-1}\ (1-t)^{c-b-1}(1-zt)^{-a} \qquad (3\ 25')$$

Using the formula[31],

$$F(a,\ b,\ c,\ z) = (1-z)^{-\varepsilon}\ F(\varepsilon,\ c-b,\ c,\ z/z-1)$$

we get,

$$(1-p^2/m^2)^{\ z}\ F(-z,\ 1+z,\ 2,\ \frac{-p^2}{m^2-p^2})$$

$$= \Gamma(-z,\ 1-z,\ 2,\ p^2/m^2)$$

which gives,

$$I(p^2,m^2,z) = -i\ \frac{\Gamma(-z)\ \Gamma(1+z)}{16\pi^2}\ m^{2z}\ F(1-z,-z,\ 2,\ p^2/m^2)$$

(3 26)

Using α-parametrization, $A(p^2,\ z)$ is written as

$$A(p^2,z) = -\frac{F(z)}{p^2}\int\frac{d^4k}{(2\pi)^4}\int_0^\infty d\alpha_1\ d\alpha_2$$

$$\exp\left(\alpha_1\left\{(p-k)^2-m^2\right\} + \alpha_2 k^2\right) - \frac{2p^2+2p\ k}{\Gamma(1-z)}\ \alpha_2^{\ -z}$$

(3 27)

Again interchanging the order of α and momentum integrations, and defining ℓ_μ as in Eq (3 19), and performing integration over ℓ using (3 20), and noting that

$$\int\frac{d^4\ell}{(2\pi)^4}\ \ell_\mu\ \exp\left[(\alpha_1+\alpha_2)\,\ell^2\right] = 0 \qquad (3\ 28)$$

We get,

$$A(p^2,z) = -i\,\frac{\Gamma(z)}{16\pi^2}\int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \;\exp\left[p^2\,\frac{\alpha_1\,\alpha_2}{\alpha_1+\alpha_2} - m^2\alpha_1\right]$$

$$\frac{\alpha_2^{-z}}{(\alpha_1+\alpha_2)^2\,\Gamma(1-z)}\left[-2 + 2\,\frac{\alpha_1}{\alpha_1+\alpha_2}\right]$$

$$= -2\Gamma(z)\left[I(p^2,m^2,z) + \frac{1}{z}\,\frac{\partial}{\partial_r{}^2}\,I(p^2,m^2,z+1)\right] \qquad (3.29)$$

which on simplification becomes,

$$A(p^2,z) = -\frac{e^2(-f^2)^z}{8\pi^2}\,(4\pi)^{-2z}\;\Gamma(-z)\,\Gamma(1-z)\;m^{2z}$$

$$\times\Big[F(1-z,\;-z,\;2,\;p^2/m^2)$$

$$-\frac{1}{2}\,(z+1)\;F(1-z,\;-z,\;3,\;p^2/m^2)\Big] \qquad (3.30)$$

and,

$$B(p^2,z) = \frac{e^2}{4\pi^2}\,(-f^2)^z\;(4\pi)^{-2z}\,\Gamma(-z)\;\Gamma(1-z)$$

$$\times\, m^{2z}\;\;F(1-z,\;z,\;2,\;p^2/m^2) \qquad (3.31)$$

Writing,

$$\Sigma(p) = \not{p}\;\;(p^2) + mB(p^2) \qquad (3.32)$$

where,

$$A(p^2) = \frac{1}{2\pi i}\int_{c_1} dz\;\Gamma(-z)\;\;A(p^2,\;z)$$

$$B(p^2) = \frac{1}{2\pi i}\int_{c_1} dz\;\;\Gamma(-z)\;\;B(p^2,\;z) \qquad (3.33)$$

The integrals (3 33) are evaluated by deforming back the contour to c and taking

$(-1)^z$ as $\cos \pi z$

so that

$$A(p^2) = -\frac{\alpha}{2\pi} \frac{1}{2\pi i} \int_{c_1'} dz \frac{(f^2 m^2)^z}{(16\pi^2)^z} \frac{[\Gamma(1-z)]^3}{z^2}$$

$$\times \cos \pi z \, [F(1-z, z, 2, p^2/m^2)$$

$$- \frac{1}{2} (z+1) F(1-z, -z, 3, p^2/m^2)] \qquad (3\ 34)$$

and,

$$B(p^2) = \frac{\alpha}{\pi} \frac{1}{2\pi i} \int_{c_1} dz \left(\frac{f^2 m^2}{16\pi^2} \right)^z \frac{[\Gamma(1-z)]^3}{z^2}$$

$$\times \cos \pi z \; F(1-z, -z, 2, p^2/m^2) \qquad (3\ 35)$$

The integrands in Eq (3 34) and (3 35) have double pole at z = 0 and triple pole at z = 1,2,3, ... Keeping only the leading terms in f (i e considering only the double pole at z = 0),

$$A(p^2) = -\frac{\alpha}{2\pi} [\ln \frac{4\pi}{fm} - \frac{3}{2} \gamma + \frac{m^2-p^2}{2p^2}$$

$$(1 + \frac{m^2+p^2}{p^2} \ln \frac{m^2-p^2}{m^2}) + 5/4] + 0 (f^2 \ln f)$$

$$(3\ 36)$$

$$B(p^2) = \frac{2\alpha}{\pi}\left[\ln\frac{4\pi}{fm} - \frac{3}{2}\gamma + \frac{m^2-p^2}{2p^2}\ln\frac{m^2-p^2}{m^2} + \frac{1}{2}\right] + O(f^2 \ln f) \tag{3.37}$$

Expanding $\sum(p)$ around $\not{p}=m$, we get,

$$\begin{aligned}\Sigma(p) &= c_1 + (\not{p}-m)\, c_2 + (\not{p}-m)^2 \Sigma_f(p) \\ &= c_1 + (\not{p}-m)\left[c_2 + (\not{p}-m)\Sigma_f(p)\right]\end{aligned} \tag{3.38}$$

and we can identify

$$\begin{aligned}c_1 = \delta m &= \Sigma(p)\big|_{\not{p}=m} \\ &= m[A(m^2) + B(m^2)]\end{aligned} \tag{3.39}$$

Using Eqs (3.36), (3.37) and (3.39), we get,

$$\delta m = \frac{3\alpha m}{2\pi}\left[\ln\frac{4\pi}{fm} - \frac{3}{2}\gamma + \frac{1}{4}\right] + O(f^2 \ln f) \tag{3.40}$$

This expression is to be compared with the result in conventional quantum electrodynamics[32] given as,

$$\delta m = \frac{3\alpha m}{2\pi}\left[\ln\frac{\Lambda}{m} + 1/4\right] \tag{3.41}$$

It can be noticed that the inverse of the minor coupling constant, f, provides a natural ultra-violet cut-off.

The expressions for $A(p^2)$ and $B(p^2)$ can be put in the following form,

$$A(p^2) = A(m^2) - \frac{\alpha}{2\pi} \frac{m^2-p^2}{2p^2} \left(1 + \frac{m^2+p^2}{p^2} \ln \frac{m^2-p^2}{m^2}\right) \tag{3.41}$$

and

$$B(p^2) = B(m^2) + \frac{2\alpha}{\pi} \frac{m^2-p^2}{p^2} \ln \frac{m^2-p^2}{m^2} \tag{3.42}$$

so that, using (3.32), (3.38), (3.41) and (3.42) we get,

$$\begin{aligned} \Sigma(p) - c_1 &= \not{p}\, A(p^2) + mB(p^2) - mA(m^2) - mB(m^2) \\ &= (\not{p}-m) \Big[A(m^2) + \frac{\alpha}{2\pi} \not{p} \frac{\not{p}+m}{2p^2} \\ &\quad \times \left(1 + \frac{m^2+p^2}{p^2} \ln \frac{m^2-p^2}{m^2}\right) \\ &\quad - \frac{2\alpha m}{\pi} \frac{\not{p}+m}{2p^2} \ln \frac{m^2-p^2}{m^2} \Big] \end{aligned} \tag{3.43}$$

From Eq. (3.38), we see that c_2 can be obtained by (3.43) putting $\not{p} = m$ in the square bracket term, so that

$$\begin{aligned} c_2 &= A(m^2) + \frac{\alpha}{2\pi} \left[\left(1 + 2 \ln \frac{m^2-p^2}{m^2}\right) - 4 \ln \frac{m^2-p^2}{m^2} \right]_{\not{p}^2=m^2} \\ &= A(m^2) + \frac{\alpha}{2\pi} \left[1 - 2 \ln \frac{m^2-p^2}{m^2} \right]\Big|_{p^2 = m^2} \end{aligned} \tag{3.44}$$

The wave-function renormalization constant for the electron is given by,

$$\begin{aligned} Z_2^{-1} &= 1 - c_2 \\ &= 1 + \frac{\alpha}{2\pi} \left[\ln \frac{4\pi}{mf} - \frac{3}{2} \gamma + \frac{5}{4} \right] - \frac{\alpha}{2\pi} \\ &\quad + \frac{\alpha}{\pi} \ln \left(\frac{m^2-p^2}{m^2} \right)\Big|_{p^2=m^2} + O(f^2 \ln f) \end{aligned}$$

Comparing (3.43) with (3.38), we get the equation,

$$c_2 + (\not p - m)\sum_f(p) = (m^2) + \frac{\alpha}{2\pi}\not p\,\frac{(\not p+m)}{2p^2}$$

$$\times\left[1 + \frac{m^2+p^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right] - \frac{2\alpha m}{\pi}$$

$$\times\frac{\not p+m}{2p^2}\ln\frac{m^2-p^2}{m^2} \qquad (3.45)$$

Using (3.44) for c_2, and multiplying both sides by $(\not p+m)$, we get,

$$(p^2-m^2)\sum_f(p) = \frac{\alpha}{2\pi}\not p\,\frac{(\not p+m)^2}{2p^2}\left[1 + \frac{m^2+p^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right]$$

$$-\frac{2\alpha m}{\pi}\,\frac{(\not p+m)^2}{2p^2}\,\ln\frac{m^2-p^2}{m^2}$$

$$-\frac{\alpha}{2\pi}(\not p+m)\left[1-2\ln\frac{m^2-p^2}{m^2}\right]_{p^2=m^2} \qquad (3.46)$$

It can be rewritten as,

$$\sum_f(p) = \frac{\alpha}{2\pi}\left[J(p^2) + (\not p+m)\,K(p^2)\right] \qquad (3.47)$$

where,

$$J(p^2) = \frac{m}{2p^2}\left[1 + \frac{m^2+p^2}{p^2}\ln\frac{m^2-p^2}{m^2} - \frac{2m}{p^2}\ln\frac{m^2-p^2}{m^2}\right] \qquad (3.48)$$

and,

$$K(p^2) = \frac{p^2+m^2}{(p^2-m^2)\,2p^2}\left[1+\frac{m^2+p^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right]$$

$$-\frac{4m}{p^2}\,\frac{m}{p^2-m^2}\,\ln\frac{m^2-p^2}{m^2} \qquad (3.49)$$

$$-\frac{1}{p^2-m^2}\left[1-2\ln\frac{m^2-p^2}{m^2}\right]_{p^2=m^2}$$

Defining $\rho = p^2/m^2$,

$$\Sigma_f(r) = \frac{\alpha}{2\pi m} \Big[\frac{1}{2(1-\rho)} \Big(1 - \frac{2-3\rho}{1-\rho} \ln\rho \Big) - \Big(\frac{\not{p}+m}{m}\Big) \Big\{ \frac{1}{2\rho(1-\rho)} \Big(\rho + \frac{-4+4\rho+\rho^2}{1-\rho} \ln\rho \Big) + \frac{2}{\rho} \ln(1-\beta) \Big|_{\beta=1} \Big\} \Big] \tag{3 53}$$

This expression is same as one obtains in conv ntional calculations[32] The logarithmic divergence in the last term in Eq (3 53) is the well known infra-red divergence

IV PHOTON SELF ENERGY

In this section we calculate the photon self energy to second order in major coupling constant e using the interaction $\mathcal{L}_{em}$ and treat the problem of gauge invariance coming in this calculation. The matrix element for this process is given by (see Fig 2)

Fig 2 Photon Self Energy

$$\Pi^{\mu\nu}(k) = ie^2 \int d^4x \; Tr\,(\gamma^\mu S_F(x)\, \gamma^\nu \; S_F(-x))$$

$$\times \exp\,(f^2 D_F(x))\, \exp\,(ik\, x) \qquad (4.1)$$

Now,

$$\exp\,[f^2\, D_F(x)] = \sum_{n=0}^{\infty} \frac{(f^2)^n}{n!}\,[D_F(x)]^n$$

$$= \frac{1}{2\pi i} \int_C dz \;\; \Gamma(-z)\, [D(x)]^z\, (-f^2)^z$$

$$(4.2)$$

where c is the same contour as used in defining Eq. (3.3). Substituting for $\exp[f^2 D_F(x)]$ from Eq. (4.2) into Eq. (4.1) and after little rearrangement we get,

$$\Pi^{\mu\nu}(k) = \frac{1}{2\pi i} \int_{c_1} dz\, \Gamma(-z)\, \Pi^{\mu\nu}(k,z) \tag{4.3}$$

where,

$$\Pi^{\mu\nu}(k,z) = ie^2 (-f^2)^z \int d^4x \, \exp(ikx)$$

$$\times \; \mathrm{Tr}\,(\gamma^\mu S_F(x)\, \gamma^\nu S_F(-x))\, [D_F(x)]^z \tag{4.4}$$

Now to be able to apply the formula (3.5), we write

$$\Pi^{\mu\nu}(k,z) = \Pi^{\mu\nu}(k,0) + \frac{1}{2\pi i} \int_{c_2} dz\, \Gamma(-z)\, \Pi^{\mu\nu}(k,z) \tag{4.5}$$

where c_2 is a contour running parallel to the imaginary axis and lying between Re z = 0 and Re z = 1. Using formula (3.5) and Eq. (3.2) and doing integration over x and one momentum integration using δ-function as in Sec. III, we get,

$$\Pi^{\mu\nu}(k,z) = \lambda(z) \int\int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\Big\{ \gamma^\mu \frac{\gamma\cdot p+m}{p^2-m^2}$$

$$\times\; \gamma^\nu \frac{\gamma\cdot(p+q-k)+m}{(p+q-k)^2-m^2} \Big\}\, (-q^2)^{z-2} \tag{4.6}$$

where,

$$k(z) = -e^2 (-f^2)^z (4\pi)^{2-2z} \frac{\Gamma(2-z)}{(z)} \qquad (4\ 7)$$

To secure convergence of momentum integrals, we shift the contour to left so as to lie between Rez = -2 and Rez = -1 In this process we encounter a simple pole at z = 0 whose contribution cancels $\Pi^{\mu\nu}(k, 0)$ in (4 5) giving,

$$\Pi^{\mu\nu}(k,z) = \frac{1}{2\pi i} \int_{c_3} dz\ \Gamma(-z)\ \Pi^{\mu\nu}(k, z) \qquad (4\ 8)$$

where $\Pi^{\mu\nu}(k, z)$ is still given by the expression (4 6) The contour c_3 is parallel to imaginary axis and lies in the region $-2 < \mathrm{Re}z < -1$ The quantity $\Pi^{\mu\nu}(k, z)$ in Eq (4 8) is essentially the same as the one appearing in ref (32), although the full $\Pi^{\mu\nu}(k)$ is different However, some of the steps in the above reference appear to be incorrect We have repeated the calculation along the same lines as in the above reference

Calculating the trace occurring in Eq (4 6), we get

$$\Pi^{\mu\nu}(k, z) = 4K(z) \iint \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (-q^2)^{z-2}$$

$$\times \frac{p^\mu(p+q-k)^\nu - p^\alpha(p+q-k)_\alpha g^{\mu\nu} + p^\nu(p+q-k)^\mu + g^{\mu\nu}m^2}{(p^2-m^2)\ [(p+q-k)^2-m^2]}$$

(4 9)

Now we write

$$\Pi^{\mu\nu}(k,z) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\, C(k^2,z) + g^{\mu\nu}\, D(k^2,z) \qquad (4.10)$$

Then we have,

$$k_\mu \Pi^{\mu\nu}(k,z) = k^\nu D(k^2,z)$$

$$= 4K(z) \int\int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \times (-q^2)^{z-2}$$

$$\times \frac{p\cdot k(p+q-k)^\nu - p^\alpha (p+q-k)_\alpha k^\nu + p^\nu (p+q-k)\cdot k + k^\nu m^2}{(p^2-m^2)\left[(p+q-k)^2 - m^2\right]}$$

(4.11)

We will first calculate the integral in Eq (4.11) for space like k and then analytically continue to the time like region as in the calculation of electron self energy. For this we use the α-parametric representation as in Sec III to get,

$$k^\nu D(k^2,z) = 4K(z) \int\int\int d\alpha_1\, d\alpha_2\, d\alpha_3 \int\int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}$$

$$\times \exp\left[\alpha_1(p^2-m^2) + \alpha_2\left[(p+q-k)^2 - m^2\right] + \alpha_3 q^2\right] \frac{\alpha_3^{-z+1}}{\Gamma(2-z)} \left[2(p\cdot k)p^\nu - (p\cdot q)k^\nu\right.$$

$$+ (p\cdot k)q^\nu + (q\cdot k)p^\nu - (p^2 k^\nu + k^2 p^\nu)$$

$$\left. + m^2 k^\nu\right] \qquad (4.12)$$

Now let,

$$\ell'_\mu = q_\mu + \frac{(p-k)_\mu\, \alpha_2}{\alpha_2+\alpha_3} \qquad (4.13)$$

using Eqs (3.20) and (3.28) we get

$$k^{\nu} D(k^2, z) = \frac{4\,(\omega)}{\Gamma(2-z)} \iiint d\alpha_1\, d\alpha_2\, d\alpha_3 \int \frac{d^4 \ell}{(2\pi)^4} \exp\,[\alpha_1(p^2-m^2)$$

$$-\alpha_2 m^2 + \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}(p-k)^2]\ \frac{i\alpha_3^{-z+1}}{16\pi^2(\alpha_2+\alpha_3)^2}\ [$$

$$\times [\{p\,(\ -k)_\alpha\, k^\nu - (p\ k)\,(p-k)^\nu - (p-k)_\alpha k^\alpha p^\nu\}$$

$$\times \frac{2}{(\alpha_2+\alpha_3)} + 2(p\ k)\ ^\nu - (\ ^2 k^\nu + k^2 p^\nu) + m^2\ ^\nu] \quad (4\ 14)$$

Making the substitution,

$$\ell''_\mu = p_\mu - k_\mu\ \frac{\alpha_2\alpha_3}{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)} \quad (4\ 15)$$

and integrating over ℓ''_μ using Eq (3 20), (3 28) and

$$\int d^4\ell\ e^{a\ell^2}\ \ell_\mu \ell_\nu = \frac{-i\ g_{\mu\nu}}{32\pi^2 a^3} \quad (4\ 16)$$

and simplifying the expression we get,

$$D(k^2, z) = -\frac{4\,\Gamma(z)}{(16\pi^2)^2\Gamma(2-z)} \iiint \frac{d\alpha_1\ d\alpha_2\ d\alpha_3}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2}\ \alpha_3^{-z+1}$$

$$\times \exp\,[\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}\ k^2 - (\alpha_1+\alpha_2)m^2]$$

$$\times [\frac{\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3} - \frac{\alpha_1\alpha_2\alpha_3^{\ 2}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2}\,k^2$$

$$+m^2] \quad (4\ 17)$$

Now to calculate the integral over α - parameters in Eq (4 16) we define x, y, z as

$$\alpha_1 = 1/x^2, \quad \alpha_2 = 1/y^2, \quad \alpha_3 = 1/z^2 \qquad (4\ 17)$$

and change to polar coordinates i e

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi$$

$$z = r \cos \theta \qquad (4\ 18)$$

Finally we get,

$$D(k^2, z) = G(z) \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^{\infty} dr\, (r)^{2z-1} \cos \phi \sin \phi$$

$$r \sin^2 \theta\, (\cos \theta)^{2z-1} [r^2 \sin^4\theta \cos^2\phi \sin^2\phi$$

$$-k^2 \sin^4\theta \cos^2\phi \sin^2\phi + m^2] \exp [k^2/r^2$$

$$- \frac{m^2}{r^2 \sin^2\theta} \cos^2\phi \sin^2\phi] \qquad (4\ 19)$$

where,

$$G(z) = \frac{-32K(z)}{\Gamma(2-z)(16\pi^2)^2} \qquad (4\ 20)$$

We first do integration over r by defining,

$$u = 1/r^2 \qquad (4\ 21)$$

and using Eq (3 23), this gives,

$$D(k^2, z) = \frac{G(z)}{2} \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta\, \Gamma(-z-1)$$

$$\times \left(\frac{m^2}{\sin^2\theta \cos^2\theta \sin^2\phi} - k^2 \right)^z$$

$$\times [\sin^4 \theta \cos^2\phi \sin^2 \phi\, k^2 - m^2(z+1-\sin^2\theta)]$$

$$\times \cos \phi \sin \phi \sin^3 \theta\, (\cos \theta)^{2z-1} \qquad (4\ 22)$$

Now to integrate over θ, we substitute $\sin^2\theta = s$ and get

$$D(k^2, z) = \frac{G(z)\,\Gamma(2-z)}{4} \int_0^{\pi/2} d\phi \cos \phi \sin \phi$$

$$\left(\frac{m^2}{\cos^2\phi \sin^2\phi} \right)^z \int_0^1 ds\left(1- \frac{\cos^2\phi \sin^2\phi k^2}{m^2} s\right)^z$$

$$\times (1-s)^{z-1} [s^{-z+3} \cos^2 \phi \sin^2\phi\, k^2 z$$

$$- m(z+1)\, s^{-z+1} + m^2 s^{z+2}] \qquad (4\ 23)$$

Now using the formula (3 25') the integration over s can be done and the result is,

$$D(k^2, z) = \frac{G(z)\,\Gamma(-1-z)}{4} (m^2)^z \int_0^{\pi/2} d\phi \sin \phi \cos \phi$$

$$\times \left(\frac{1}{\cos^2\phi \sin^2\phi}\right)^z \left[\frac{\Gamma(z)\,\Gamma(4-z)}{\Gamma(4)}\right.$$

$$\times\ {}_2F_1 \left(-z, 4-z, 4, \frac{\cos^2\phi \sin^2\phi k^2}{m^2} \right)$$

$$x \cos^2\phi \sin^2\phi \, k^2 \, z - m^2(z+1) \frac{\Gamma(z)\Gamma(3-z)}{\Gamma(3)}$$

$$ {}_2F_1 \left(-z,\ 2-z,\ 2,\ \frac{\cos^2\phi \sin^2\phi}{m^2} k^2\right)$$

$$ + m^2 \frac{\Gamma(z)\Gamma(3-z)}{\Gamma(3)} \ {}_2F_1 \left(-z, 3-z,\ 3,\ \frac{\cos^2\phi \sin^2\phi \, k^2}{m^2}\right) \Big] \tag{4.24}$$

To perform the remaining integral over ϕ we substitute

$$\sin^2 2\phi = t \tag{4.25}$$

and the result is,

$$D(k^2, z) = \frac{1}{16} \sigma(z)\, \Gamma(z)\, \Gamma(-z-1)\, (4m^2)^z \int_0^1 dt\, (1-t)^{-1/2}$$

$$x\ t^{-z} \Big[z \frac{\Gamma(4-z)}{4\,\Gamma(4)}\, tk^2 \ {}_2F_1 \left(-z,\ 4-z,\ 4,\ \frac{tk^2}{4m^2}\right)$$

$$+ m^2 \frac{\Gamma(3-z)}{\Gamma(3)} \ {}_2F_1 \left(-z,\ 3-z,\ 3,\ \frac{tk^2}{4m^2}\right)$$

$$- m^2(z+1) \frac{\Gamma(2-z)}{\Gamma(2)} \ {}_2F_1 \left(-z,\ 2-z,\ 2,\ \frac{tk^2}{4m^2}\right)\Big] \tag{4.26}$$

Now we can integrate over t using the formula[9]

$$\int_0^1 dx\, (1-x)^{\mu-1}\, x^{\nu-1} \ {}_2F_1 (a_1, a_2,\ b,\ \alpha x)$$

$$= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \ {}_3F_2 (\nu,\ a_1, a_2,\ \mu+\nu,\ \nu,\ b,\ \alpha)$$

and obtain,

$$D(k^2,z) = \frac{1}{16} G(z)\,\Gamma(z)\,\Gamma(-1-z)\,(4m^2)^z$$

$$\times [\, z k^2 \frac{\Gamma(1/2)\,\Gamma(2-z)}{\Gamma(5/2-z)} \frac{\Gamma(4-z)}{4\Gamma(4)}\; {}_3F_2\,(2-z, -z,$$

$$4-z,\; 5/2-z,\; 4,\; k^2/4m^2) + m^2$$

$$\times \frac{\Gamma(3-z)\,\Gamma(1/2)\,\Gamma(1-z)}{\Gamma(3)\,\Gamma(3/2-z)}\; {}_3F_2(1-z,\; -z,$$

$$3-z,\; 3/2-z,\; 3,\; k^2/4m^2) - m^2\,(z+1)$$

$$\times \frac{\Gamma(2-z)\,\Gamma(1/2)\,\Gamma(1-z)}{\Gamma(2)\,\Gamma(3/2-z)}\; {}_3F_2\,(1-z,\; -z,$$

$$2-z,\; 3/2-z,\; 2,\; k^2/4m^2)] \qquad (4.27)$$

In the above equation we change ${}_3F_2$ functions in terms of Meijer's G-functions[31] and after some algebra we get,

$$D(k^2,z) = \frac{1}{64} G(z)\,\Gamma(z)\,\Gamma(1/2)\,(-k^2)^{z+1}$$

$$[\, G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} -1,\; 1,\; -3 \\ -z, -3/2,\; -3-z \end{matrix}\right)$$

$$+ G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} -1,\; 1,\; -3 \\ -1-z, -3/2, -3-z \end{matrix}\right)$$

$$+ G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} -1,\; 0,\; -2 \\ -1-z,\; -3/2,\; -2 \end{matrix}\right)] \qquad (4.28)$$

The expression in square brackets can be simplified[31] to get,

$$-3z\, G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} -2,\ -1,\ 1 \\ -z-1,\ -z-3,\ -3/2 \end{matrix}\right) \qquad (4\ 29)$$

Using the relation

$$(-k^2)^{z+1}\ G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} -2,\ -1,\ 1 \\ -z-1,\ -z-3,\ -3/2 \end{matrix}\right)$$

$$= (4m^2)^{z+1}\ G^{13}_{33}\left(-\frac{k^2}{4m^2}\,\middle|\, \begin{matrix} z-1,\ z,\ z+2 \\ 0,\ -2,\ z-1/2 \end{matrix}\right) \qquad (4\ 30)$$

and converting the G functions into F function and substituting for G(z) we get,

$$D(k^2,\ z) = -\frac{5e^2\ (-f^2)^z}{3\pi^{2+2z}}\ \frac{(m^2)^{z+1}}{4^z}$$

$$\times\ \frac{\Gamma(2-z)\,\Gamma(1-z)\,\Gamma(-z)}{(1+z)\,\Gamma(3/2-z)\,\Gamma(3)}\ {}_3F_2\,(2-z,\ 1-z,$$

$$-1-z,\ 3/2-z,\ 3,\ k^2/4m^2) \qquad (4\ 31)$$

Now,

$$C(k^2,z) = [g_{\mu\nu}\,\Pi^{\mu\nu}(k,z) - 4D(k^2,z)]/3k^2 \qquad (4\ 32)$$

We have,

$$g_{\mu\nu}\Pi^{\mu\nu} = 4K(z)\iint \frac{d^4q\ d^4p}{(2\pi)^8}\,(-q^2)^{z-2}$$

$$\times\left[\frac{-2p^{\alpha}(p+q-k)_{\alpha} + 4m^2}{(p^2-m^2)\left\{(p+q-k)^2-m^2\right\}}\right] \qquad (4\ 33)$$

using once again α-parametric representation and evaluating integrals over momenta, we get,

$$g_{\mu\nu}\Pi^{\mu\nu} = -\frac{4\Gamma(z)}{(16\pi^2)^2}\iiint d\alpha_1\, d\alpha_2\, d\alpha_3$$

$$\times \exp\left[\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}k^2 - (\alpha_1+\alpha_2)\, m^2\right]$$

$$\times \frac{1}{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^2}\left[\frac{4\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}\right.$$

$$\left. + k^2\frac{2\alpha_1\alpha_2\alpha_3{}^2}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2} + m^2\right]\frac{\alpha_3{}^{1-z}}{\Gamma(2-z)} \qquad (4\ 34)$$

Now using Eq (4 34) and Eq (4 16) We get from Eq (4 32),

$$C(k^2, z) = -\frac{8\Gamma(z)}{\Gamma(2-z)(16\pi^2)^2}\iiint d\alpha_1\, d\alpha_2\, d\alpha_3$$

$$\times\ \alpha_3{}^{3-z}\ \frac{\alpha_1\alpha_2}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^4}$$

$$\times \exp\left[\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}\ k^2-(\alpha_1+\alpha_2)m^2\right] \quad (4\ 35)$$

The integration over α parameters can be carried out as has been done for $D(k^2,z)$ earlier The end result is

$$C(k^2, z) = \frac{e^2}{16\pi^{2+2z}}(-f^2)^z\ \frac{(m^2)^z}{4^z}\ \frac{\Gamma(1/2)}{\Gamma(4)}$$

$$\times\ \frac{\Gamma(-z)\Gamma(4-z)\Gamma(2-z)}{\Gamma(5/2 - z)}\ {}_3F_2\ (2-z,\ -z,\ 4-z,$$

$$5/2 -z,\ 4,\ k^2/4m^2) \qquad (4\ 36)$$

Now from Eqs (4 8) and (4 10) we have

$$\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\, C(k^2) + \ldots D(k^2) \qquad (4\ 37)$$

where

$$C(k^2) = \frac{1}{2\pi i} \int_{C_3} dz\, \Gamma(-z)\, C(k^2, z) \qquad (4\ 38)$$

$$D(k^2) = \frac{1}{2\pi i} \int_{C_3} dz\, \Gamma(-z)\, D(k^2, z) \qquad (4\ 39)$$

From Eqs (4 36) and (4 31) it is seen that the integrand in (4 38) has double pole at $z = 0$ and integrand in (4 39) has a simple poles at $z = 0$ and at $z = -1$ Calculating contributions of these poles we get, upto terms of order f^0,

$$C(k^2) = -\frac{\alpha}{3\pi}\Big[-2 \log \frac{2\pi}{mf} + \frac{2\lambda-1}{2\lambda}\left(\frac{\lambda+1}{\lambda}\right)^{1/2} \times \log \frac{\sqrt{1+\lambda} + \sqrt{\lambda}}{\sqrt{1+\lambda} - \sqrt{\lambda}} + \frac{1}{\lambda} - \frac{1}{6} + 2\gamma - \log 4\Big] + O(f^2 \log f) \qquad (4\ 40)$$

and

$$D(k^2) = -\frac{8\pi\alpha}{f^2} - \frac{3\alpha}{2\pi}\left(m^2 - \frac{k^2}{9}\right) + O(f^2 \log f) \qquad (4\ 41)$$

where

$$\lambda = -k^2/4m^2$$

This gives for the photon wave function renormalisation constant

$$z_3 = 1 - C(0)$$

$$= 1 - \frac{\alpha}{3\pi}\left[\log\frac{4\pi^2}{m^2 f^2} + \frac{1}{6} - 3\gamma + \log 4 - \frac{5}{3}\right] + O(f^2 \log f) \qquad (4.42)$$

Now we write $C(k^2)$ as,

$$C(k^2) = C(0) + k^2\, \pi^f(k^2)$$

identifying $\pi^f(k^2)$ as,

$$\pi^f(k^2) = -\frac{\alpha}{3\pi}\left[\left(\frac{2\lambda-1}{2\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)^{1/2}\log\frac{\sqrt{1+\lambda}+\sqrt{\lambda}}{\sqrt{1+\lambda}-\sqrt{\lambda}} - \frac{5}{3} + \frac{1}{\lambda}\right] + O(f^2 \log f) \qquad (4.43)$$

This result agrees with the standard result in quantum electrodynamics[33]

Gauge Invariance

In the above calculation $D(k^2)$ is not zero which implies lack of gauge invariance. We note that $1/f^2$ term in $D(k^2)$ is reminiscent of the quadratic divergence in this quantity encountered in naive perturbation theoretic calculation[33] in conventional quantum electrodynamics. In the latter case a careful calculation employing a well defined gauge invariant current operator[34,35] or appropriate gauge invariant regularization[29] ensures $D = 0$.

In our formalism also, we should modify the current operator, because the product of field operators at the same space time point is singular [illegible], therefore, we replace $\bar{\psi}\gamma^{\mu}\psi$ in $\mathcal{L}_{em}$ by strictly gauge invariant operator[34,35]

$$\bar{\psi}(x + \varepsilon/2)\,\gamma^{\mu}\,\psi(x - \varepsilon/2)\exp\left[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2} d\beta^{\nu} A_{\nu}(\beta)\right] \tag{4.44}$$

which gives rise to the additional interaction,

$$\mathcal{L}'_{em} = ie^{2}\,\bar{\psi}(x + \varepsilon/2)\,\gamma^{\mu}\,\psi(x - \varepsilon/2)\,A_{\mu}(x)\,\exp\left[f\sigma(x)\right]$$

$$\times\int_{x-\varepsilon/2}^{x+\varepsilon/2} d\beta^{\nu} A_{\nu}(\beta) \tag{4.45}$$

To obtain the additional contribution $\Pi'_{\mu\nu}(k^{2})$ to the polarisation tensor, due to the above interaction, we must express the first order S-matrix due to $\mathcal{L}'_{em}$ in the form[36]

$$S_{1} = i\int d^{4}x\,\mathcal{L}'_{em}(x)$$

$$= -i\int d^{4}k\, A^{\mu}(k)\,\Pi_{\mu\nu}(k)\,A^{\nu}(k) + \ldots \tag{4.46}$$

where,

$$A_{\mu}(k) = \frac{1}{(2\pi)^{2}}\int d^{4}x\, e^{ik\cdot x}\, A_{\mu}(x) \tag{4.47}$$

β-integration in Eq. (4.45) can be carried out along a straight line path $\beta^{\nu} = x^{\nu} + \frac{1}{2}s\,\varepsilon^{\nu}$, $-1 \le s \le 1$

Noting

$$\langle 0 | \bar{\psi}(x+\varepsilon/2)\, \gamma_\mu \psi(x-\varepsilon/2) | 0 \rangle = -\mathrm{Tr}(\gamma_\mu S(-\varepsilon)) \qquad (4.48)$$

we obtain[5] after a straightforward calculation,

$$\Pi'_{\mu\nu}(k) = ie^2 \varepsilon_\nu \mathrm{Tr}[\gamma_\mu S(-\varepsilon)] \frac{2}{k\,\varepsilon} \sin(k\,\varepsilon/2) \qquad (4.49)$$

This gives in the limit $\varepsilon^\nu \to 0$, a quadratically divergent contribution to $D(k^2)$ which cancels the usual quadratic divergence in $D(k^2)$. Calling the contribution to $D(k^2)$ from (4.49) as $D'(k^2)$ and using Eq. (4.10) and (4.8) we have,

$$[D(k^2)]_{total} = D'(k^2) + \frac{1}{2\pi i} \int_{C_1} dz\, \Gamma(-z)\, D(k^2, z)$$

$$= D'(k^2) + I(k^2, 0) + \frac{1}{2\pi i} \int_{C_2} dz\, \Gamma(-z) D(k^2, z) \qquad (4.50)$$

$D(k^2, 0)$ is the usual quadratically divergent quantity and, as mentioned above

$$\lim_{\varepsilon^\nu \to 0} [D'(k^2) + D(k^2, 0)] = 0 \qquad (4.51)$$

Thus giving,

$$[D(k^2)]_{total} = \frac{1}{2\pi i} \int_{C_2} dz\, \Gamma(-z)\, D(k^2, z) \qquad (4.52)$$

sum of zero, one, two, graviton exchanges. The diagram in Fig. (3) contains modified electron-electron-graviton vertex which is separately shown in Fig. 4.

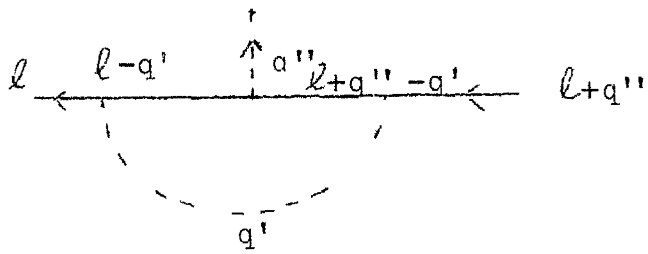


Fig. 4 Modified Electron-Electron-Graviton Vertex

The matrix element for this diagram is given by,

$$M(\ell, q'') = \int_{\alpha-i\infty}^{\alpha+i\infty} dz_1 \, h(z_1) \int d^4q' \, \gamma_\nu (2\ell-q')$$

$$\times \frac{1}{\gamma(\ell-q')-m} \, \gamma(2\ell-2q'+q'')$$

$$\times \frac{1}{\gamma(\ell+q''-q')-m} \, \gamma(2\ell+2q''-q')$$

$$\times (-q'^2)^{z_1-2} \qquad (4.53)$$

where $-1 < \alpha < 0$. The function $h(z_1)$ includes the z-dependent factors other than $(-q^2)^{z-2}$ in Eq. (3.5) and various constants. The integral over q' in Eq. (4.53) is obviously convergent without shifting the contour c'.

Next we examine the high momentum behaviour of $M(\ell, q'')$. Putting

$$|q''| = s^2$$

and,

$$-1 < \alpha < -3/4 \tag{4.54}$$

one can easily see that

$$\lim_{s \to \infty} M(\ell, s, \hat{q}'') \lesssim s^{-1} (\log s)^{n_1} \tag{4.55}$$

where n_1 is a non-negative integer.

Next we determine the asymptotic behaviour with respect to ℓ. Setting

$$|\ell| = t^2$$

$$\lim_{\substack{t \to \infty \\ q'' \text{ fixed}}} M(t, \hat{\ell}, q'') \lesssim t^{-1} (\log t)^{n_2} \tag{4.56}$$

n_2 being some other non-negative integer and where $\hat{\ell}_\mu = \ell_\mu / |\ell|$. Finally we find behaviour of $M(\ell, q'')$ when both ℓ and q'' tend to infinity.

Let,

$$q'' = \omega^2 Q'' \qquad \ell = \omega^2 \ell''$$

$$q' = \omega^2 Q''' \qquad m = \omega^2 m_o$$

Then,

$$M(\zeta, q'') \underset{\omega\to\infty}{\sim} \omega^{-1} (\log \omega)^{1} \qquad (4\ 57)$$

Now, the matrix element for the diagram in Fig (3) is given by,

$$k^2 D(k^2) = k^{\mu}\, {}^{\nu}\pi_{\mu\nu}(k^2)$$

$$= \int_{\beta-i\infty}^{\beta+i\infty} dz \int_{\alpha-i\infty}^{\alpha+i\infty} dz_2 \int d^4 p\, d^4 q''\, d^4 q$$

$$\times \mathrm{Tr}\,[\gamma k \frac{1}{\gamma (p+q-k)-m} M(p-k+q, q'')$$

$$\times \frac{1}{\gamma (p+q''-k+q)-m} \gamma k \frac{1}{\gamma (p+q'')-m}$$

$$\times M(p+q'', -q'') \frac{1}{\gamma p-m}] (-q^2)^{z-2}$$

$$\times (-q^2)^{z_2-1}, \quad 0 < \beta < 1 \qquad (4\ 58)$$

Now by power counting we can see that q'' and q integrations are convergent. To see convergence of p-integration we let $|p| = u^2$ and then by power counting, we can see that u - integration is also convergent. Hence $D(k^2) \lesssim O(f^2 \log f)$. This completes the proof of gauge invariance of $\pi_{\mu\nu}(k)$ upto order f^0.

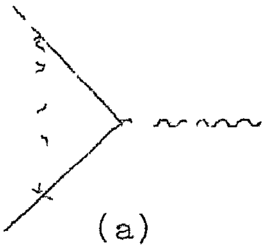
7 VERTEX PART

In this section we calculate the vertex part by using L_{em}. The Feynmann Diagram is shown in Figure 5(a). The quantity $\Lambda^{\nu}(p', p)$ in lowest order is given by

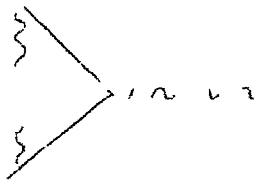
$$\Lambda^{\nu}(p', p) = e^2 \iint d^4x \, d^4y \; \gamma^{\alpha} \, S_F(x) \exp [f^2 D_F(x)] \gamma^{\nu}$$

$$\times S_F(y) \exp [f^2 D_F(y)] \; \gamma_{\alpha} \, D_F(x+y)$$

$$\times \exp [f^2 D_F(x+y)] \; e^{ip'x} \, e^{ip\,y} \qquad (5.1)$$



(a)



(b)

Fig 5 Lowest order vertex correction diagram

It turns out that the term represented by diagram 5(b) is convergent and inclusion of other two superpropagators as shown in diagram 5(a) can alter the result in order $f^2 \ln f$. Therefore, to order $(f)^0$, it is sufficient to consider the diagram 5(b), so that

$$\Lambda^{\nu}(p',p) = e^2 \iint d^4x\, d^4y\, \gamma^{\alpha} S_F(x)\, \gamma^{\nu} S_F(y)\, \gamma_{\alpha}\, D_F(x+y)$$

$$\times \exp\left[f^2 D_F(x+y)\right] \exp\left[ip'x\right] \exp\left[ipy\right] \tag{5 2}$$

Proceeding as in Section III, we get,

$$\Lambda^{\nu}(p',p) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dz\, \Gamma(-z)\, \Lambda^{\nu}(p',p,z)$$

$$-1 < \alpha < 0 \tag{5 3}$$

where,

$$\Lambda^{\nu}(p',p,z) = F(z)\int \frac{d^4q}{(2\pi)}\, \gamma^{\alpha}\, \frac{(\not{p}'-\not{q})+m}{(p'-q)^2-m^2}$$

$$\times\, \gamma^{\nu}\frac{(\not{p}-\not{q})+m}{(p-q)^2-m^2}\, \gamma_{\alpha}\, (-q^2)^{z-1} \tag{5 4}$$

where F(z) is given by Eq (3 8)

Now we shall show that $\Lambda^{\nu}(p',p)$ given by Eq (5 3) satisfies the Ward Identity[6] Multiplying Eq (5 4) by $(p-p')_{\nu}$ and summing over ν, we get

$$(p-p')_{\nu}\,\Lambda^{\nu}(p',p,z) = F(z)\int \frac{d^4q}{(2\pi)^4}$$

$$\times\, \gamma^{\alpha}\left[\frac{1}{\not{p}'-\not{q}-m} - \frac{1}{\not{p}-\not{q}-m}\right]\gamma_{\alpha}$$

$$\times\, (-q^2)^{z-1}$$

$$= \Sigma(p',z) - \Sigma(p,z) \tag{5 5}$$

Hence,

$$(p-p')_{\nu} \Lambda^{\nu}(p',p) = \Sigma(p') - \Sigma(p) \qquad (5.6)$$

which completes the proof of Ward's identity to order f^0, since the inclusion of the other two superpropagators as shown in Fig 5(a), can alter the result in the order $f^2 \ln f$. The direct calculations with vertex are very complicated. However, we can easily relate the vertex renormalization factor and the traditional finite part left after the renormalization to the quantities already calculated and to some quantities in conventional quantum electrodynamics. It is to be noticed that for the contour lying between -1 and 0, the integral in Eq (5.4) is convergent. We now write,

$$\Lambda^{\nu}(p', p) = L\,\gamma^{\nu} + \Lambda_f^{\nu}(p',p) \qquad (5.6)$$

where,

$$L\gamma^{\nu} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dz\ \Gamma(-z)\ \Lambda^{\nu}(p,p,z)\Big|_{\not p=m} \qquad (5.7)$$

From the Ward identity Eq (5.5), we have $L = -c_2$ where c_2 is the quantity calculated in Sec III. Also,

$$\Lambda_f^{\nu}(p',p) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dz\ \Gamma(-z)\ \Lambda_f^{\nu}(p',p,z) \qquad (5.8)$$

We can write,

$$\Lambda_{+}^{\nu}(p',p,z) = \int d^4q \, [h^{\nu}(q,p',p,z) - h^{\nu}(q,p',p,z)\Big|_{\not p=m}] \times (-q^2)^{-1} \qquad (5.9)$$

where,

$$h^{\nu}(q,p',p,z) = \Gamma(z)\, \gamma^{\alpha} \frac{1}{(\not p'-\not q-m)} \gamma^{\nu} \frac{1}{\not p-\not q-m} \gamma_{\alpha} \qquad (5.10)$$

The integration in Eq (5.9) over q for z = 0 is finite. The quantity $\Lambda_{f}^{\nu}(p',p,0)$ is obviously the finite part in conventional quantum electrodynamics.

Now,

$$\Lambda_{f}^{\nu}(p',p) = \Lambda_{f}^{\nu}(p',p,0) + \int_{c_2} dz \, \Gamma(-z) \times \Lambda_{+}^{\nu}(p',p,z) \qquad (5.11)$$

where c_2 is the contour described in Sec. III. By shifting the contour c_2 such that $0 < \mathrm{Re}\,z < 1/2$, the integral over q in Eq (5.11), is seen to be convergent. Therefore, the integral in Eq (5.11) only gives the terms of order $f^2 \ln f$ and higher. Moreover, the addition of the other two superpropagators in Fig. 5(a) affects the result to order $f^2 \ln f$ or higher.

We have, therefore, shown that upto zeroth order in f the vertex renormalization constant as given by the Ward identity and is infinite. The finite part left after renormalization is the same as in the conventional quantum electrodynamics.

CENTRAL LIBRARY
53972
Acc. No.

VI PROOF OF FINITENESS OF THE THEORY TO ALL ORDERS

In this section, we present the proof of the suppression of ultraviolet divergences for the theory formulated to all orders in e

We consider a general graph in arbitrary order of e It has between any two vertices,

(a) zero, one or two fermion lines,

(b) zero or one photon line,

(c) one superpropagator

Working in Feynman gauge, the photon propagator whenever present can be absorbed in the superpropagator as we did for electron self energy calculation For the time being we ignore the term $\mathcal{L}'$ in the Lagrangian and consider matrix elements due to $\mathcal{L}_{em}$ only In the contribution of a given diagram there are following integrations (a) over superpropagator momenta q_{ij} (b) over momentum of each internal electron line and (c) over the complex variables z_{ij} Also there are, in the integrand, the δ-functions giving energy momentum conservation at each vertex and Gamma functions coming from Sommerfeld Watson transform and the Fourier transforms of super-propagators

(i.e. from use of Gelfand Shilov formula for $L_F^z(x)$). Thus, ignoring certain non-essential factors the matrix element for a general diagram can be written in the symbolic form as

$$M \sim \int \prod_{i<j} dz_{ij}\, d^4q_{ij}\, \Gamma(-z_{ij}) \left(-\frac{f^2}{16\pi^2}\right)^{z_{ij}} \frac{\Gamma(2-z_{ij}-\alpha_{ij})}{\Gamma(z_{ij}+\alpha_{ij})} (-q_{ij}^2)^{z_{ij}+\alpha_{ij}-2} \prod_a d^4p_a \times \frac{1}{\gamma\, p_a - m} \prod_k \delta(P_k - \sum p - \sum q) \qquad (6.1)$$

where the symbols α_{ij} represent number of photon lines between vertices i and j (0 or 1) and P_k represents the sum of external momenta at the k-th vertex. The contours of integration lie parallel to the imaginary axis with $-1 < \mathrm{Re}\, z_{ij} \le 0$.

For formula (6.1) it has been assumed that whenever necessary the contours have been shifted to right to satisfy Gelfand Shilov condition i.e. $0 < \mathrm{Re}\,(z_{ij} + \alpha_{ij}) < 2$, and after the Gelfand Shilov formula has been substituted the contours have been taken back (as was done in the photon self energy calculation). This causes no problems.

Now a simple observation simplifies the proof of finiteness tremendously. The observation is as follows.

The variables z_{ij} appear with positive sign in the exponent of q_{ij}^2 and in Γ- functions in the denominator and with a negative sign in Γ-functions in the numerator. This observation leads us to the conclusion that contours for z_{ij} can be shifted to the left to secure convergence of q_{ij} integrations and no new singularities are introduced.

Now in Eq (6.1) some of the integrations over momenta can be done trivially by δ-functions. After these integrations, we are either left with only some superpropagator momentum integrations as in the case of electron self energy (See III) and vertex part (See V), which are convergent by the above argument or there may remain integrations over some electron momenta. In a situation of the latter type at least one of the electron propagators in the integrand will involve a linear combination of the electron loop momentum and at least one of the superpropagator momenta as in the photon self energy case in Section IV. Now since we can make the q_{ij} integrations as much convergent as we like, giving asymptotic behavior for large p as $\frac{1}{|p|^{\beta}}[\log|p|]^n$ with β becoming greater and greater for shifting the relevant z-contour more and more to the left, thus ensuring the convergence of p-integrations.

Now performing momentum integrations, we get,

$$M \sim \int \prod_{i<j} dz_{ij} \frac{\Gamma(-z_{ij})\Gamma(2 z_{ij} - \alpha_{ij})}{\Gamma(z_{ij} + \alpha_{ij})} \left(-\frac{f^2}{16\pi^2}\right)^{z_{ij}} f(P,z) \qquad (6\ 2)$$

where $f(P, z)$ is some function of the external momenta and z_{ij}'s obtained after the q and r integrations Now it can be easily seen that $f(P, z)$ is a bounded function of z_{ij}'s on their contours This can be seen as follows Symbolically, we can write,

$$f(P, z) = \int d^4p\, d^4q \quad g(P, p, q, z)$$

therefore, $|f(P,z)| \leq \int d^4p\, d^4q\, |g(P,p,q,z)|$

Now it should be obvious from the above discussion that q and p integration are in fact absolutely convergent This proves the statement made above

For the asymptotic behavior of Γ-functions we have

$$\Gamma(az+b) \sim \sqrt{2\pi}\, e^{-az}(az)^{az+b-1/2}, \quad \arg z < \pi, \quad a > 0 \qquad (6\ 3)$$

Writing $z = x+iy = re^{i\theta}$ we have,

$$|\Gamma(az+b)| \sim (\text{constant})e^{-xa}\,(ar)^{ax+b-1/2}\, e^{-y\theta a} \qquad (6\ 4)$$

Equation (6 4) shows that for large z, $\Gamma(z)$ decreases very fast except along the negative real axis (note that y and θ always have the same sign) Since in Eq (6 2) there are more Γ-functions in the numerator than in the denominator, the z-integrations are convergent

The above arguments apply to any graph and its subgraphs The proof of ultraviolet convergence of any graph computed with the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{em}$ is complete

Introduction of $\mathcal{L}'$ makes the following modifications

(a) Two new types of vertices are introduced - those involving electron and σ-lines only and those involving σ lines only

(b) The coupling $\mathcal{L}'$ will give rise to some momentum factors at the vertices

The new superpropagators arising from (a) will again give rise to the same kind of z dependence as before so that our convergence argument will go through

The point (b) can also be taken care of by shifting the z-contours to the left

We must also consider the fact that the contours for the superpropagator lines arising from the couplings in $\mathcal{L}'$ will lie in the region $0 < \text{Re } z_{ij} < 1$ so that when one of these contours is shifted to the left, the factor $\Gamma(-z_{ij})$ gives singularity at $z_{ij} = 0$ The simplest way to see that this does not create any problem is the following We can write,

$$\mathcal{L}' = \mathcal{L}'' + \mathcal{L}'''$$

where,

$$\mathcal{L}'' = \tfrac{1}{2}\,[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (\partial_{\mu}\bar{\psi})\,\gamma^{\mu}\psi]\; e^{f\sigma(x)} + \tfrac{1}{2}\,\partial^{\mu}\sigma\,\partial_{\mu}\sigma\;\; e^{-2f\sigma(x)}$$

and,

$$\mathcal{L}''' = -\frac{1}{2}\left[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\,\gamma^{\mu}\psi\right] - \frac{1}{2}\,\partial^{\mu}\sigma\,\partial_{\mu}\sigma$$

Now if $\mathcal{L}''$ is used instead of $\mathcal{L}'$ the contours for the superpropagators lie in the region $-1 < \mathrm{Re}\, z_{ij} < 0$ and hence there is no problem. The coupling $\mathcal{L}'''$ introduces only some momentum factors in the internal electron and σ- lines of the diagrams arising from the interaction $\mathcal{L}_{cm} + \mathcal{L}''$. These can be taken care of as point (b) above.

This completes the proof of ultraviolet finiteness of the theory in any arbitrary order.

VII GRAVITATIONAL SELF ENERGY OF ELECTRON

In this section we calculate the electron self energy due to interaction L_1' given in Eq (2 33b) in lowest order The contribution $\Sigma_g(p)$ from this interaction term (diagram is shown in Figure 6) is given by,

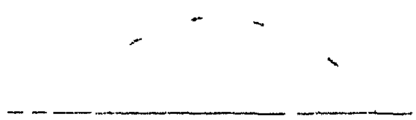


Fig 6 Gravitational contribution to self energy of electron

$$\Sigma_g(p) = \frac{i}{4}\int d^4x \Big\{ \Big[\gamma^\mu \frac{\partial}{\partial x^\mu} S_F(x)\, \gamma^\nu (-ip_\nu) + \gamma^\mu \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} S_F(x)\, \gamma^\nu - p_\mu\, \gamma^\mu S_F(x)\, \gamma^\nu p_\nu - ip_\mu\, \gamma^\mu \frac{\partial S_F(x)}{\partial x^\nu}\, \gamma^\nu \Big] \big[\exp (f^2 D_F(x))-1\big] e^{ip\,x}\Big\} \qquad (7\ 1)$$

Now,

$$[\exp (f^2 D_F(x)) - 1] = \sum_{n=1}^{\infty} \frac{(f^2)^n}{n!} [D_F(x)]^n$$

$$= \frac{1}{\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} dz\ \ \Gamma(-z)\ [D_F(z)]^z$$

$$0 < \alpha < 1 \qquad (7\ 2)$$

and in subsequent discuss on the subscript g refers to the gravitational contribution From Eq (3 2) and (3 5), we get,

$$\Sigma_g(p) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \Gamma(-z)\,(-f^2)^z\,\Sigma_g(p,z)\,dz \qquad (7\ 3)$$

$$0 < \beta < 1$$

where,

$$\Sigma_g(p,z) = -\frac{1}{4}F'(z)\int\frac{d^4k}{(2\pi)^4}\Big[\gamma(p-k)\,\frac{\gamma(p-k)+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma\,p$$

$$+\gamma(p-k)\,\frac{\gamma(p-k)+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma\,(p-k)$$

$$+\gamma\,p\,\frac{\gamma(p-k)+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma\,p$$

$$+\gamma\,p\,\frac{\gamma(p-k)+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma\,(p-k)\Big]\,(-k^2)^{z-2} \qquad (7\ 4)$$

where,

$$F'(z) = (4\pi)^{2-2z}\,\frac{\Gamma(2-z)}{\Gamma(z)} \qquad (7\ 5)$$

After simplifying the expression (7 4), we get,

$$\Sigma_g(p,z) = -\frac{1}{4}F'(z)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(p-k)^2-m^2+i\varepsilon}$$

$$\times\,[4\gamma\,p(\gamma\,(p-k)+m)\gamma\,p-2\gamma\,k(\gamma\,(p-k)+m)\gamma\,p$$

$$+\gamma\,k(\gamma\,(p-k)+m)\gamma\,k-2\gamma\,p\,(\gamma\,(p-k)+m)\gamma\,k]$$

$$\times\,(-k^2)^{z-2} \qquad (7\ 6)$$

As in Section III, we write

$$\Sigma_g(p,z) = \not{p}\, A_g(p^2,z) + m\, B_g(p^2,z) \tag{7.7}$$

where,

$$A_g(p^2,z) = \frac{1}{4p^2}\,\mathrm{tr}\,[\not{p}\,\Sigma_g(p,z)]$$

$$B_g(p^2,z) = \frac{1}{4m}\,\mathrm{tr}\,[\Sigma_g(p,z)] \tag{7.8}$$

Evaluating trace in Eq (7.8) and after simplifying we get

$$A_g(p^2,z) = -\frac{iF'(z)}{4p^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(p-k)^2-m^2+i\varepsilon}$$

$$[4p^4-3(p\cdot k)p^2 + 2(p\cdot k)^2 + 3p^2k^2-k^2(p\cdot k)]$$

$$\times\,(-k^2)^{z-2} \tag{7.9}$$

$$= -i\,\frac{F'(z)}{4p^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(p-k)^2-m^2+i\varepsilon}$$

$$\times\,[4p^4-3(p\cdot k)p^2 + 2(p\cdot k)^2]\,(-k^2)^{z-2}$$

$$-\,\frac{iF'(z)}{4p^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(p-k)^2-m^2+i\varepsilon}$$

$$\times\,[-3p^2+p\cdot k](-k^2)^{z-1} \tag{7.10}$$

Using α-parametric representation, we get

$$A_g(p^2,z) = \frac{iF'(z)}{4p^2}\int\frac{d^4k}{(2\pi)^4}\int_0^\infty\int_0^\infty d\alpha_1\, d\alpha_2$$

$$\times \exp\left[\alpha_1\left\{(p-k)^2-m^2\right\}+\alpha_2 k^2\right]$$

$$\times \frac{\alpha_2^{1-z}}{\Gamma(2-z)}\left[4k^4-8k^2(p\cdot k)+2(p\cdot k)^2\right]$$

$$+\frac{iF'(z)}{4p^2}\int\frac{d^4k}{(2\pi)^4}\int_0^\infty\int_0^\infty d\alpha_1\, d\alpha_2$$

$$\times \exp\left[\alpha_1\left\{(p-k)^2-m^2\right\}+\alpha_2 k^2\right]$$

$$\times \frac{\alpha_2^{-z}}{\Gamma(1-z)}\left[-3p^2+p\cdot k\right] \qquad (7.11)$$

To integrate over momentum k, we define a variable ℓ_μ by

$$k_\mu = \ell_\mu + \frac{\alpha_1}{\alpha_1+\alpha_2}\, p_\mu$$

and integrating over the momentum ℓ, we get

$$A_g(p^2,z) = \frac{iF'(z)}{4}\int_0^\infty\int_0^\infty d\alpha_1\, d\alpha_2 \exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2-\alpha_1 m^2\right]$$

$$\times\frac{\alpha_2^{1-z}}{\Gamma(2-z)}\frac{1}{16\pi^2(\alpha_1+\alpha_2)^2}\left[4p^2-8p^2\frac{\alpha_1}{\alpha_1+\alpha_2}\right.$$

$$\left.-\frac{1}{\alpha_1+\alpha_2}+2\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2\right]+\frac{iF'(z)}{4}\int_0^\infty\int_0^\infty d\alpha_1\, d\alpha_2$$

$$\times \exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2-\alpha_1 m^2\right]\frac{\alpha_2^{1-z}}{\Gamma(1-z)}$$

$$\times\frac{1}{16\pi^2(\alpha_1+\alpha_2)^2}\left[-3+\frac{\alpha_1}{\alpha_1+\alpha_2}\right] \qquad (7.12)$$

Now, expressing the integration over α-parameters in terms of $I(p^2,m^2,z)$ given in Eq (3 21), we get

$$
\begin{aligned}
I_g(p^2,z) = \frac{F'(z)}{4} \Big[& -4p^2 I(p^2,m^2,z-1) + \frac{8p^2}{(1-z)} \frac{\partial}{\partial p^2} I(p^2,m^2,z) - \frac{2p^2}{(1-z)(-z)} \frac{\partial^2}{\partial (p^2)^2} \\
& \times I(p^2, m^2, z+1) \Big] + \frac{F'(z)}{4} \Big[3I(p^2,m^2,z) \\
& - \frac{1}{(-z)} \frac{\partial^2}{\partial (p^2)^2} I(p^2,m^2,z+1) \Big] + \frac{F'(z)}{4} \int_0^\infty \int_0^\infty \\
& \times \frac{d\alpha_1 \, d\alpha_2}{(\alpha_1+\alpha_2)^3} \exp\left[\frac{\alpha_1 \alpha_2}{\alpha_1+\alpha_2} p^2 - \alpha_1 m^2 \right] \\
& \times \frac{\alpha_2^{1-z}}{16\pi^2 \Gamma(2-z)}
\end{aligned}
\qquad (7\ 13)
$$

The integral in the above expressions have been evaluated in the Appendix B and has the value,

$$
\int\int \frac{d\alpha_1 \, d\alpha_2}{(\alpha_1 + \alpha_2)^3} \exp\left[\frac{\alpha_1 \alpha_2}{\alpha_1+\alpha_2} p^2 - \alpha_1 m^2 \right] \alpha_2^{1-z}
$$

$$
= (m^2)^z \frac{\Gamma(1-z) \Gamma(2-z) \Gamma(1+z)}{\Gamma(3)}
$$

$$
\times F(-z, 2-z, 3, p^2/m^2)
$$

Using the expression for $I(p^2,m^2,z)$ from Eq (3 24), substituting the value of $F'(z)$ from Eq (7 5) and after some

simplification, we get,

$$A_g(p^2, z) = (4\pi)^{-2z} \, \underline{\Gamma(2-z)\,\Gamma(1-z)} \; m^{2z}$$

$$\times \Big[- \frac{4p^2}{m^2} F(2-z,\ 1-z,\ 2,\ p^2/m^2)$$

$$+ \frac{4p^2}{m^2} z \ F(2-z,\ 1-z,\ 3,\ p^2/m^2)$$

$$- \frac{p^2}{3m^2} (1+z)\, z\, F(2-z,\ 1-z,\ 4,\ p^2/m^2)$$

$$- 3F(1-z,\ 1-z,\ 2,\ p^2/m^2)$$

$$+ \frac{1}{2} (1+z)\, F(1-z,\ -z,\ 3\ \ p^2/m^2)$$

$$- \frac{1}{2} F(-z,\ 2-z,\ 3,\ p^2/m^2) \Big] \qquad (7\ 14)$$

From Eq (7 8), we get,

$$B_g(p^2,z) = -\frac{1}{4} F'(z) \int \frac{d^4 k}{(2\pi)^4} \ \frac{(4p^2-4p\ k+k^2)}{(p-k)^2-m^2} \ (-k^2)^{z-2} \qquad (7\ 15)$$

Again using α-parametrization we get,

$$B_g(p^2,z) = \frac{1}{4} F'(z) \iint d\alpha_1 \, d\alpha_2 \int \frac{d^4 k}{(2\pi)^4}$$

$$\times \exp\left[\alpha_1 \left\{(p-k)^2-m^2\right\} + \alpha_2 k^2\right] \frac{\alpha_2^{\,1-z}}{\Gamma(2-z)}$$

$$\times \left[4p^2 - 4p\ k + k^2\right] \qquad (7\ 16)$$

The integration over k is again carried out by defining,

$$k_\mu = \ell_\mu + \frac{\alpha_1}{\alpha_1+\alpha_2} p_\mu$$

giving,

$$B_g(p^2,z) = \frac{1}{4} F'(z) \int d\alpha_1 \int d\alpha_2 \quad \exp\left[\frac{\alpha_1 \quad 2}{\alpha_1+\alpha_2} p^2 - \alpha_1 m^2\right]$$

$$x \frac{\alpha_2^{1-z}}{\Gamma(2-z)} \quad \frac{1}{16\pi^2(\alpha_1+\alpha_2)^2}$$

$$x \left[4p^2 - \frac{4\alpha_1}{\alpha_1+\alpha_2} p^2 - \frac{2}{\alpha_1+\alpha_2} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2} p^2\right] \tag{7 17}$$

In term of $I(p^2,m^2,z)$, it becomes,

$$B_g(p^2,z) = \frac{iF'(z)}{4} \left[-4p^2 I(p^2,m^2,z-1)\right.$$

$$+ \frac{4p^2}{1-z}\frac{\partial}{\partial p^2} I(p^2,m^2,z) - \frac{p^2}{(1-z)(-z)}$$

$$x \left.\frac{\partial^2}{\partial(p^2)^2} I(p^2,m^2,z+1)\right] + \frac{2F'(z)}{4} \int_0^\infty \int_0^\infty$$

$$x \frac{d\alpha_1 \, d\alpha_2}{16\pi^2(\alpha_1+\alpha_2)^3} \quad \exp\left[\frac{\alpha_1 \, \alpha_2}{\alpha_1+\alpha_2} p^2 - \alpha_1 m^2\right]$$

$$x \frac{\alpha_2^{1-z}}{\Gamma(2-z)} \tag{7 18}$$

substituting for I (p^2,m^2,z) and after a little algebra, we obtain,

$$B_o(p^2,z) = \frac{(4\pi)^{-2z}}{4} \Gamma(2-z)\, \Gamma(1-z)\, m^{-z}$$

$$\Big[-\frac{4p^2}{m^2} F(2-z,\ 1-z,\ 2,\ p^2/m^2)$$

$$+\frac{2p^2}{m^2} z\, F(2-z,\ 1-z,\ 3,\ p^2/m^2)$$

$$-\frac{p^2}{6m^2}(1+z)\, z\, F(2-z,\ 1-z,\ 4,\ p^2/m^2)$$

$$-F(-z,\ 2-z,\ 3,\ p^2/m^2)\Big] \qquad (7\ 19)$$

Now from expressions (7 3) and (7 7) we get,

$$\Sigma_g(p^2) = \not{p}\, A_g(p^2) + m\, B_g(p^2) \qquad (7\ 20)$$

where,

$$A_g(p^2) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dz\ \Gamma(-z)\,(-f^2)^z A_g(p^2,z)$$

and,

$$B_g(p^2) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dz\ \Gamma(-z)\,(-f^2)^z B_g(p^2,z)$$

$$0 < \alpha < 1 \qquad (7\ 21)$$

where,

$$A_g(p^2,z) \quad \text{and} \quad B_g(p^2,z)$$

are given by Eq (7 14) and (7 19)

Folding the contour enclosing real axis and considering the leading contribution to $A_g(p^2)$ and $B_g(p^2)$ from the double pole at $z = 1$, get

$$A_g(p^2) = -2\pi i \Big(\text{Res of integrand in Eq (7 21) at } z = 1 \Big)$$

$$= \frac{d}{dz} \frac{[\Gamma(2-z)]^3}{4} \quad (-p^2m^2/16\pi^2)^z$$

$$\times \Big[-\frac{4p^2}{m^2 z} F\,(2-z,\ 1-z,\ 2,\ p^2/m^2)$$

$$+ \frac{4p^2}{m^2} F\,(2-z,\ 1-z,\ 3,\ p^2/m^2)$$

$$- \frac{p^2}{3m^2} (1+z) \quad (2-z,\ 1-z,\ 4,\ p^2/m^2)$$

$$- \frac{3}{z} F\,(1-z,\ -z,\ 2,\ p^2/m^2)$$

$$+ \frac{1+z}{2z} \quad (1-z,\ -z,\ 3,\ p^2/m^2)$$

$$- \frac{1}{2z} F(-z,\ 2-z,\ 3,\ p^2/m^2)\Big] \Big|_{z=1} \qquad (7\ 22)$$

$$= \frac{1}{2} \left(\frac{mf}{4\pi}\right)^2 \ln \frac{4\pi}{mf} \left[-\frac{1}{2} \frac{p^2}{m^2} - \frac{5}{2} \right]$$

$$- \frac{3}{4} \gamma \left(\frac{mf}{4\pi}\right)^2 \left[-\frac{1}{2} \frac{p^2}{m^2} - \frac{5}{2} \right]$$

$$- \frac{1}{4} \left(\frac{mf}{4\pi}\right)^2 \psi_1 \quad (p^2/m^2)$$

where the function $\psi_1(x)$ is given by

$$\psi_1(x) = (1-x) \frac{8x^2 + 7x+1}{2x^2} \ln (1-x)$$

$$+ \frac{5}{4} x + \frac{23}{4} + \frac{1}{2x} \qquad (7\ 23)$$

Similarly,

$$B_g(p^2) = -2\pi i \{\text{Res of Integrand in } (7\ 21), \text{ at } z = 1\}$$

$$= \frac{1}{2} (\frac{mf}{4\pi})^2 \ln \frac{4\pi}{mf} \left[- \frac{2p^2}{m^2} - 1 \right]$$

$$- \frac{3\gamma}{4} (\frac{mf}{4\pi})^2 \left[\frac{-2p^2}{m^2} - 1 \right]$$

$$- \frac{1}{4} (\frac{mf}{4\pi})^2 \psi_2 (p^2/m^2) \qquad (7\ 24)$$

where,

$$\psi_2(x) = \frac{2(1+x)(1-x)}{x} \ln (1-x) + \frac{9}{2} x + 3 \qquad (7\ 25)$$

Finally

$$(\delta m)_g = m[A_g(m^2) + B_g(m^2)]$$

$$= m (\frac{mf}{4\pi})^2 [-3 \ln \frac{4\pi}{mf} + \frac{9}{2} \gamma - \frac{15}{4}] \qquad (7\ 26)$$

Note The contour in Eq (7 3) has been shifted to the left to secure convergence of the k-integration and then shifted back to the region $0 < \text{Re } z < 1$

VIII CALCULATION OF $\pi^+ - \pi^0$ MASS DIFFERENCE

As an application of the theory, here we present calculation of $\pi^+-\pi^0$ mass difference to lowest order in coupling constant e. Proceeding as in sec. II, the conformal invariant Lagrangian for a charged scalar field Φ and a neutral scalar field Φ^0 can be written as,

$$= D^\mu \Phi^\dagger D_\mu \Phi + \frac{1}{2} D^\mu \Phi^0 D_\mu \Phi^0 - \mu^2 \Phi^\dagger \Phi \exp(-2f\sigma(x))$$

$$- \frac{1}{2}\mu^2 \Phi^{02} \exp(-2f\sigma(x)) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2}\partial^\mu\sigma\,\partial_\mu\sigma$$

$$\times \exp(-2f\sigma(x)) \qquad (8.1)$$

where,

$$D_\mu \Phi = \partial_\mu \Phi + f\partial_\mu\sigma\,\Phi + i e A_\mu \Phi$$

and $$D_\mu \Phi^0 = \partial_\mu \Phi^0 + f\partial_\mu\sigma\,\Phi^0 \qquad (8.2)$$

Applying the transformation,

$$\varphi = \Phi \exp(f\sigma(x))$$

and $$\varphi^0 = \Phi^0 \exp(f\sigma(x)) \qquad (8.3)$$

the Lagrangian (8.1) becomes,

$$L = L_0 + L_{em} + L' \qquad (8.4)$$

where,

$$
\begin{aligned}
\mathcal{L}_0 &= \partial^\mu \Phi^+ \partial_\mu \Phi + \tfrac{1}{2} \partial^\mu \Phi^0 \partial_\mu \Phi^0 - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} \partial^\mu \sigma \partial_\mu \sigma \\
&\quad - \mu^2 \Phi^+ \Phi^- - \tfrac{1}{2} \mu^2 \Phi^0 \Phi^0 \\
\mathcal{L}_{em} &= -ie \, [\Phi^+ \partial^\mu \Phi - \partial^\mu \Phi^+ \Phi] \, A_\mu \exp\,(2f\sigma(x)) \\
&\quad + e^2 \Phi^+ \Phi A^\mu A_\mu \exp\,(2f\sigma(x)) \\
\mathcal{L}' &= \partial^\mu \Phi^+ \partial_\mu \Phi (\exp\,(2f\sigma) - 1) + 2f[\partial^\mu \Phi^+ \Phi \\
&\quad + \Phi^+ \partial^\mu \Phi] \, \partial_\mu \sigma \exp\,(2f\sigma) + 4f^2 \partial^\mu \sigma \partial_\mu \sigma \Phi^+ \Phi \exp(2f\sigma) \\
&\quad + f[\partial_\mu \Phi^0 \Phi^0 + \Phi^0 \partial_\mu \Phi^0] \, \partial^\mu \sigma \exp\,(2f\sigma) + 2f^2 \partial^\mu \sigma \\
&\quad \times \partial_\mu \sigma \Phi^0 \Phi^0 \exp\,(2f\sigma) + \tfrac{1}{2} \partial^\mu \sigma \partial_\mu \sigma \, [\exp\,(2f\sigma) - 1]
\end{aligned}
\tag{8.5}
$$

where normal ordering for the above Lagrangian is understood

The lowest order contribution to $\Pi(p)$ [see Fig 7] is given by,

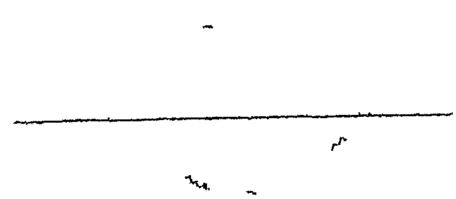


Fig 7 Diagram for self energy of Pion Here the solid lines represents the charged pion

$$\prod(p) = -ie^2 \int d^4x \, [-ip_\mu \frac{\partial \Delta_F(x)}{\partial x_\mu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\Delta_F(x)$$

$$- \mu^2 \Delta_F(x) - ip_\mu \frac{\partial \Delta_F(x)}{\partial x_\mu}] \, e^{ipx}$$

$$= -ie^2 \int d^4x \, [-2ip_\mu \frac{\partial \Delta_F(x)}{\partial x_\mu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\Delta_F(x)$$

$$- p^2 \Delta_F(x)] \, e^{ipx} \, D_F(x) \exp(4f^2 D_F(x)) \qquad (8\ 6)$$

where,

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\mu^2+i\varepsilon} \exp(-ikx) \qquad (8\ 7)$$

and $D_F(x)$ is given by Eq (3 2)

Since,

$$D_F(x) \exp (4f^2 D_F(x)) = \sum_{n=0}^{\infty} \frac{(4f^2)^n}{n!} [D_F(x)]^{n+1}$$

$$= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dz \, (-4f^2)^z \, \Gamma(-z) \, [D_F(x)]^{z+1} \qquad (8\ 8)$$

$$-1 < \alpha < 0$$

(by Sommerfeld-Watson transformation)

using expressions for $\Delta_F(x)$, D_F and Gelfand and Shilov formula (3 5), and performing x-integration and the trivial momentum integration, we get,

$$\prod(p) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(-z) \, \prod(p,z) \, dz \qquad (8\ 9)$$

where,

$$\Pi(p,z) = F_2(z) \int \frac{d^4k}{(2\pi)^4} \frac{(-k^2)^{z-1}}{(p-k)^2 - m^2 + i\varepsilon} (4p^2 - 4p\,k + k^2) \tag{8.10}$$

and where,

$$F_2(z) = ie^2 (4\pi)^{-2z} \frac{\Gamma(1-z)}{\Gamma(1+z)} (-4f^2)^z \tag{8.11}$$

Using α-parametric representation, we get,

$$\Pi(p, z) = -F_2(z) \int_0^\infty \int_0^\infty d\alpha_1 \, d\alpha_2 \int \frac{d^4k}{(2\pi)^4}$$

$$\times \exp\left[\alpha_1 \left\{(p-k)^2 - m^2\right\} + \alpha_2 k^2\right] \frac{\alpha_2^{-z}}{\Gamma(1-z)}$$

$$\times [4p^2 - 4p\,k + k^2]$$

$$= -i \frac{F_2(z)}{16\pi^2} \int_0^\infty \int_0^\infty \frac{d\alpha_1 \, d\alpha_2}{(\alpha_1+\alpha_2)^2} \exp\left\{\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)} p^2 - \alpha_1 m^2\right\}$$

$$\frac{\alpha_2^{-z}}{\Gamma(1-z)} \left[4p^2 - 4p^2 \frac{\alpha_1}{\alpha_1+\alpha_2} - \frac{2}{\alpha_1+\alpha_2} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2} p^2\right] \tag{8.12}$$

where in obtaining the last form we have used Eq (3.20), (3.28) and (4.16) after performing k-integration by redefinition,

$$k_\mu = l_\mu + \frac{\alpha_1}{\alpha_1 + \alpha_2} p_\mu \tag{8.13}$$

Eq (8.12) can be written in terms of the integrals $I(p^2, m^2, z)$ i.e.

$$\Pi(p, z) = F_2(z)\,\Big[4p^2 I(p^2,m^2,z) + \frac{4i^2}{z}\,\frac{\partial}{\partial p^2}\,I(p^2,m^2,z+1)$$

$$+ \frac{1}{(-z)(-1-z)}\,\frac{\partial^2}{\partial (p^2)^2}\,I(p^2,m^2,z+2)\Big]$$

$$+ \frac{2iF_2(z)}{16\pi^2}\int_0^\infty\int_0^\infty \frac{d\alpha_1\, d\alpha_2}{(\alpha_1+\alpha_2)^3}$$

$$\exp\Big[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\,p^2 - \alpha_1 m^2\Big]\,\frac{\alpha_2^{-z}}{\Gamma(1-z)} \qquad (8.14)$$

Expression (3.26) for $I(p^2,m^2,z)$ and the value of the integral in the last term from Appendix B can be substituted in the above equation, leading to,

$$\Pi(p, z) = \frac{e^2\,(4\pi)^{-2z}}{16\pi^2}\,\Gamma(-z)\,\Gamma(1-z)\,(-4f^2)^z$$

$$(m^2)^{z+1}\Big[\frac{4p^2}{m^2}\,F(1-z, -z, 2, p^2/m^2)$$

$$-\frac{2p^2}{m^2}\,(1+z)\,F(1-z, -z, 3, p^2/m^2)$$

$$+\frac{p^2}{6m^2}\,(1+z)\,(2+z)\,F(1-z,-z, 4, p^2/m^2)$$

$$+ 2F(-1-z, 1-z, 3, p^2/m^2)\Big] \qquad (8.15)$$

$\Pi(p)$ is given as the contour integral of $\Pi(p,z)$ and can be evaluated by folding the contour back to enclose the real-axis Considering only the leading terms

$$\Pi(p) = -2\pi i \left(\text{Res of Integrand in Eq (8 9)}\Big|_{z=0} \right)$$

$$= -\frac{e^2}{16\pi^2} m^2 \frac{d}{dz} [(-\frac{f^2 m^2}{4\pi^2})^z \{\Gamma(1-z)\}^3 h(p,z)]_{z=0} \qquad (8\ 16)$$

where h(p,z) is the expression in square brackets in Eq (8 15),

$$= \frac{\alpha}{4\pi} \mu_\pi^2 [(\frac{5}{3} \frac{p^2}{m^2} + 2) \ln \frac{4\pi^2}{m^2 f^2} - 3\gamma (\frac{5}{3} \frac{p^2}{m^2} + 2)$$

$$- \xi (p^2/m^2)] \qquad (8\ 17)$$

where,

$$\xi(x) = (1-x) \frac{-5x^2 - 8x + 1}{3x^2} \ln(1-x)$$

$$- \frac{17}{9} x - \frac{17}{6} + \frac{1}{3x} \qquad (8\ 18)$$

This gives,

$$\delta\mu^2 = \frac{\alpha}{4\pi} \mu^2 [\frac{11}{3} \ln (\frac{4\pi^2}{m^2 f^2}) - 11 \gamma + \frac{79}{18}]$$

The numerical value of $\pi^+ - \pi^0$ mass difference using the value of minor coupling constant f given by Eq (2 44) comes 13 6 MeV which is much higher than the experimental value 4 6 MeV We have considered only the minimal electro-magnetic interactions of pions and have neglected strong

interaction effects The cut-off provided by the scalar gravity is very high, however strong interactions are expected to provide an effective cut off at a smaller value which will reduce the $\pi^+ - \pi^0$ mass difference

IX CONCLUDING REMARKS

In the previous sections we have seen that scalar gravity plays a regularizing role for electromagnetic interactions, the inverse of minor coupling constant provides the ultraviolet cut-off. Some advantages of this regularization procedure are as follows

(1) Our prescription is not an adhoc one but has a general principle behind it, namely conformal invariance, which is more appealing and beautiful than any adhoc prescription

(2) Our regularization method is universal since any Poincare invariant Lagrangian can be turned into a conformal invariant one by a well defined procedure. Thus, we can apply this method to other field theories also like the Yukawa interaction and the four fermion interaction

The Yang-Mills theories appear to be an exception since they are already conformal invariant (i.e. without the introduction of σ-field) if one uses canonical scale dimension. However, there is no compulsion to use canonical scale dimensions. In fact, in Section II, the fields obtained after the field transformation have non-canonical scale dimensions. Now if we take non-canonical weights for Yang-Mills fields,

exponentials of the field $\sigma(\ )$ will have to be introduced with kinetic energy terms and interaction term and we shall have the theory regularized

(3) Our prescription is much simpler than the tensor gravity in which calculations are very complicated Moreover, tensor gravity has its own divergence problems, whereas in our scheme, the theory is finite including all effects of gravity in all orders

(4) Since our theory has in-built conformal invariance, hence the amplitudes are expected to have scaling properties which are observed experimentally

It should be interesting to study the implications of the present model at high energies For the time being we have postponed it for future investigations

APPENDIX A

In this Appendix we present proof of Equations (2 22) and (2 25) We have,

$$\psi'(x') = \left|\det \frac{\partial x'}{\partial x}\right|^{\ell_\psi/4} D(\Lambda(x))\,\psi(x) \qquad (A\ 1)$$

$$\frac{\partial \psi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\, J^{-\ell_\psi}\, D(\Lambda(x))\left[\partial_\nu \psi - \frac{\ell_\psi}{J}\partial_\nu J\psi + D^{-1}\partial_\nu D\psi\right] \qquad (A\ 2)$$

where,

$$J = \left|\det \frac{\partial x'}{\partial x}\right|^{-1/4} \qquad (A\ 3)$$

Now $D^{-1}\partial_\nu D$ can be expressed as linear combination of the generators of the group

$$D^{-1}\partial_\nu D = k_\nu^{\rho\sigma}\, S_{\rho\sigma} \qquad (A\ 4)$$

and,

$$\Lambda^{-1}\partial_\nu \Lambda = k_\nu^{\rho\sigma}\, t_{\rho\sigma} \qquad (A\ 5)$$

where,

$$(t_{\rho\sigma})^\mu_\nu = i(\delta^\mu_\rho\, g_{\sigma\nu} - \delta^\mu_\sigma\, g_{\rho\nu}) \qquad (A\ 6)$$

$$\mathrm{Tr}(t_{\rho\lambda}\, t_{\sigma\varphi}) = 2(g_{\rho\sigma}\, g_{\lambda\varphi} - g_{\rho\varphi}\, g_{\sigma\lambda}) \qquad (A\ 7)$$

$$\mathrm{Tr}(\Lambda^{-1}\partial_\nu\Lambda\, t_{\rho\lambda}) = 4k_{\nu,\rho\lambda} \qquad (A\ 8)$$

Now let,

$$\Lambda = JA \qquad (A\ 9)$$

where,

$$A^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \qquad (A\ 10)$$

So,

$$\Lambda^{-1}\partial_\nu \Lambda = J^{-1} A^{-1} (A \partial_\nu J + J \partial_\nu A) \qquad (A\ 11)$$

$$= J^{-1}\partial_\nu J + A^{-1}\partial_\nu A \qquad (A\ 11)$$

$$(A^{-1}\partial_\nu A)^{\alpha}_{\mu} = (A^{-1})^{\alpha}_{\gamma}\partial_\nu A^{\gamma}_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\gamma}} \frac{\partial^2 x'^{\gamma}}{\partial x^{\nu}\partial x^{\mu}} \qquad (A\ 12)$$

Now using Equations (A 5) and (A 8), we have

$$4k_{\nu,\rho\lambda} = \mathrm{Tr}\,(\Lambda^{-1}\partial_\nu \Lambda\ t_{\rho\lambda})$$

$$= \mathrm{Tr}\,(A^{-1}\partial_\nu A\ t_{\rho\lambda})$$

$$= (A^{-1}\partial_\nu A)^{\alpha}_{\beta}\,(t_{\rho\lambda})^{\beta}_{\alpha}$$

$$= i[-(A^{-1}\partial_\nu A)_{\rho\lambda} + (A^{-1}\partial_\nu A)_{\lambda\rho}]$$

$$= i[\,g_{\alpha\lambda} \frac{\partial x^{\alpha}}{\partial x'^{\gamma}} \frac{\partial^2 x'^{\gamma}}{\partial x^{\nu}\partial x^{\rho}} - g_{\alpha\rho} \frac{\partial x^{\alpha}}{\partial x'^{\gamma}} \frac{\partial^2 x'^{\gamma}}{\partial x^{\nu}\,\partial x^{\lambda}}] \qquad (A\ 13)$$

Now for special conformal transformations,

$$x'^{\mu} = \frac{x^{\mu} + \beta^{\mu}x^2}{1+2\beta\, x + \beta^2 x^2} = \frac{1}{2}\frac{\partial}{\partial \beta_{\mu}} \log J(x,\ \beta) \qquad (A\ 14)$$

Similarly,

$$x^{\mu} = \frac{x'^{\mu} - \beta^{\mu}x'^{2}}{1-2\beta\, x' + \beta^{2}x'^{2}} = -\frac{1}{2}\frac{\partial}{\partial\beta_{\mu}} J(x', \beta) \qquad (A\ 15)$$

After straightforward calculation and some simplification one obtains,

$$\frac{\partial x^{\mu}}{\partial x'^{\rho}} = \frac{1}{2}\frac{\partial^{2} J(x, \beta)}{\partial x_{\mu}\, \partial \beta^{\rho}} - \frac{1}{2}\frac{\partial J(x, \beta)}{\partial x_{\mu}} \times \frac{\partial J(x, \beta)}{\partial \beta^{\rho}} \qquad (A\ 16)$$

$$\frac{\partial^{2} x'^{\rho}}{\partial x^{\nu}\partial x^{\sigma}} = \frac{1}{2}\frac{\partial^{3}(\log J(x, \beta))}{\partial x^{\nu}\, \partial x^{\sigma}\, \partial \beta_{\rho}}$$

$$= \frac{1}{2}\Big[\frac{2}{J^{3}}\frac{\partial J}{\partial x^{\nu}}\frac{\partial J}{\partial x^{\sigma}}\frac{\partial J}{\partial \beta_{\rho}} - \frac{1}{J^{2}}\frac{\partial^{2} J}{\partial x^{\nu}\partial x^{\sigma}}$$

$$\times\frac{\partial J}{\partial \beta_{\rho}} - \frac{1}{J^{2}}\frac{\partial J}{\partial x^{\sigma}}\frac{\partial^{2} J}{\partial x^{\nu}\,\partial \beta_{\rho}} - \frac{1}{J^{2}}\frac{\partial J}{\partial x^{\nu}}$$

$$\times\frac{\partial^{2} J}{\partial x^{\sigma}\,\partial \beta_{\rho}} + \frac{1}{J}\frac{\partial^{2} J}{\partial x^{\nu}\,\partial x^{\sigma}\,\partial \beta_{\rho}}\Big] \qquad (A\ 17)$$

Then after some straightforward calculation and some simplification, we get,

$$\frac{\partial x^{\mu}}{\partial x'^{\rho}}\,\frac{\partial^{2} x'^{\rho}}{\partial x^{\nu}\,\partial x^{\sigma}} = \frac{1}{J}\Big[g_{\nu\sigma}\frac{\partial J}{\partial x_{\mu}} - \delta^{\mu}_{\nu}\frac{\partial J}{\partial x^{\sigma}} - \delta^{\mu}_{\sigma}\frac{\partial J}{\partial x^{\nu}}\Big] \qquad (A\ 18)$$

Now Eq (A 4) becomes,

$$D^{-1}\partial_{\nu} D = -\frac{1}{2J}\, S_{\rho\nu}\Big(g^{\lambda}_{\nu}\frac{\partial J}{\partial x_{\rho}} - g^{\rho}_{\nu}\frac{\partial J}{\partial x_{\lambda}}\Big)$$

$$= -\frac{1}{J}\, S_{\nu\rho}\frac{\partial J}{\partial x_{\rho}} = i\, S_{\nu\rho}\, \partial^{\rho}\log J \qquad (A\ 19)$$

Equation (A 2) then becomes,

$$\frac{\partial \psi'(x')}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} J^{-\ell_\psi} D(\Lambda(x)) [\partial_\nu \psi - (\ell_\psi g_{\nu\rho} - i S_{\nu\rho}) \partial^\rho J]$$

$$= \frac{\partial x^{\nu}}{\partial x'^{\mu}} \left|\det \frac{\partial x'}{\partial x}\right|^{\ell_\psi/4} D(\Lambda(x)) [\partial_\nu \psi +$$

$$\frac{1}{4} (\ell_\psi g_{\nu\rho} - i S_{\nu\rho}) \partial^\rho (\log\left|\det \frac{\partial x'}{\partial x}\right|) \psi] \quad \text{(A 20)}$$

Now to prove Eq (2 25), we proceed as follows,

$$\Delta_\mu \psi(x) = \partial_\mu \psi(x) - f(\ell_\psi g_{\mu\nu} - i S_{\mu\nu})(\partial^\nu \sigma)\psi$$

$$\partial_\mu \psi'(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \left|\det \frac{\partial x'}{\partial x}\right|^{\ell_\psi/4} D(\Lambda(x)) [\partial_\nu \psi +$$

$$\frac{1}{4} (\ell_\psi g_{\nu\rho} - i S_{\nu\rho}) \partial^\rho \log \left|\det \frac{\partial x'}{\partial x}\right| \psi]$$

$$\sigma'(x') = \sigma(x) + \frac{1}{4f} \log \left|\det \frac{\partial x'}{\partial x}\right|$$

$$(\partial^\nu \sigma)'(x') = g^{\nu\alpha} (\partial_\alpha \sigma)'(x')$$

$$= g^{\nu\alpha} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} [\partial_\beta \sigma + \frac{1}{4f} \partial_\beta \log \left|\det \frac{\partial x'}{\partial x}\right|]$$

Therefore,

$$
\begin{aligned}
(\Delta_\mu \psi)'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \left|\det \frac{\partial x'}{\partial x}\right|^{\ell\psi/4} D(\Lambda) \, [\, \partial_\nu \psi + \\
\frac{1}{4} \left(\ell\psi \, g_{\nu\rho} - i\sigma_{\nu\rho} \right) \partial^\rho \log \left|\det \frac{\partial x'}{\partial x}\right| \psi \,] \\
- f \left(\ell\psi \, g_{\mu\nu} - i\sigma_{\mu\nu} \right) g^{\nu\alpha} \, \frac{\partial x^\rho}{\partial x'^\alpha} \\
\times [\, \partial_\rho \sigma + \frac{1}{4f} \partial_\beta \log \left|\det \frac{\partial x'}{\partial x}\right| \,] \\
\times \left|\det \frac{\partial x'}{\partial x}\right|^{\ell\psi/4} D(\Lambda(x)) \, \psi(x)
\end{aligned}
$$

Now, noting,

$$
\begin{aligned}
D^{-1} \sigma_{\mu\nu} D &= g_{\mu\alpha} \, g_{\nu\beta} \, D^{-1} \sigma^{\alpha\beta} D \\
&= g_{\mu\alpha} \, g_{\nu\beta} \, \Lambda^\alpha_\gamma \, \Lambda^\beta_\rho \, \sigma^\gamma
\end{aligned}
$$

and simplifying we get the desired relation,

$$
\begin{aligned}
(\Delta_\mu \psi)'(x') = \left|\det \frac{\partial x'}{\partial x}\right|^{(\ell\psi - 1)/4} \Lambda^\nu_\mu(x) \, D(\Lambda(x)) \\
\times \Delta_\nu \psi
\end{aligned}
$$

APPENDIX B

In this appendix, we present a calculation of the integral occuring in Eq (7.13),

$$R(p,\ z) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{1}{(\alpha_1 + \alpha_2)^3}$$

$$\times \exp \left[\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2 - \alpha_1 m^2 \right] \alpha_2^{-z} \qquad \text{(B 1)}$$

Let,

$$\alpha_1 = 1/x^2 \quad , \quad \alpha_2 = 1/y^2 \qquad \text{(B 2)}$$

$$d\alpha_1\ d\alpha_2 = \frac{4}{x^3 y^3}\ dx\ dy \qquad \text{(B 3)}$$

We have,

$$R(p,\ z) = 4 \int_0^\infty \int_0^\infty dx\ dy \quad \frac{x^3 y^3}{(x^2+y^2)^3} (y^2)^z$$

$$\exp\ (p^2/(x^2+y^2) - m^2/x^2) \qquad \text{(B 4)}$$

Now let,

$$x = \gamma \ \cos\theta \quad , \quad y = \gamma \ \sin\theta \qquad \text{(B 5)}$$

$$dx\ dy = \gamma \ d\theta\ d\gamma \qquad \text{(B 6)}$$

This gives,

$$R(p,z) = 4 \int_0^\infty \int_0^{\pi/2} dr\, d\theta\, r^{2z+1} \cos^3\theta \sin^3\theta$$

$$\times (\sin^2\theta)^z \exp\left(\frac{1}{r^2}\left(p^2 - \frac{m^2}{\cos^2\theta}\right)\right] \quad (B\ 7)$$

Let

$$1/r^2 = \mu$$

$$dr = -\frac{u^{-3/2}}{2} du$$

so that

$$\int_0^\infty dr\, r^{2z+1} \exp\left[\frac{1}{r^2}(p^2 - m^2/\cos^2\theta)\right]$$

$$= \frac{1}{2}\int_0^\infty du\, u^{-z-2} \exp\left[(p^2 - m^2/\cos^2\theta) u\right]$$

$$= \frac{1}{2}\left(\frac{m^2}{\cos^2\theta} - p^2\right)^{z+1} \Gamma(-1-z) \qquad (B\ 8)$$

Therefore,

$$R(p, z) = 2\Gamma(-1-z) \int_0^{\pi/2} d\theta \cos^3\theta \sin^3\theta (\sin^2\theta)^z$$

$$\times (\cos^2\theta)^{-z-1} (m^2)^{z+1} \left[1 - \frac{p^2}{m^2}\cos^2\theta\right]^{z+1}$$

(B 9)

Writing $\cos^2\theta = x$

so that,

$$-2\sin\theta \cos\theta = dx$$

then,

$$R(1, z) = \int_0^1 dx\, x^{-z} (1-x)^{z+1} \left[1 - \frac{p^2}{m^2} x\right]^{z+1}$$

$$\times (m^2)^{z+1} \quad \Gamma(-1-z)$$

$$= (m^2)^{z+1} \quad \Gamma(-1-z) \quad \frac{\Gamma(1-z)\, \Gamma(2+z)}{\Gamma(3)}$$

$$\times F(-1-z, 1-z, 3, p^2/m^2)$$

REFERENCES

1 I Waller, Zeits Phys 62, 673 (1930)

2 V Weisskopff, Zeits, Phys 89, 27 (1934), Phys Rev 56, 72 (1939)

3 F J Dyson, Phys Rev 75, 1736 (1949)

4 J Glimm and A Jaffe, Comm Math Phys 11, 9 (1968)

5 S Okubo, Progr Theo Phys 11, 80 (1951)
G V Efimov, Sov Phys JETP 17, 1417 (1963), Nuovo Cimento 32, 1046 (1969), Nucl Phys 74, 657 (1965)

E S Fradkin, Nucl Phys 49, 624 (1963),
M K Volkov, Ann Phys (N Y) 49, 202 (1968),
R Delbourgo, A Salam and J Strathdee, Phys Rev 187, 1999 (1969)

6 F Budini and G Calucci, Nuovo Cimento 70A, 419 (1970)

7 S Deser, Rev Mod Phys 29, 417 (1957),
B S Dewitt, Phys Rev 162, 1195 (1967)

8 R Delbourgo, A Salam and J Strathdee, Nuovo Cimento Letters 2, 354 (1969)

9 C J Isham, A Salam and J Strathdee, Phys Rev D3, 1805 (1971)

10 For a review, see A Salam in 'Developments in High Energy Physics', Proceedings of the International School of Physics 'Enrico Fermi', Academic Press, (1972)

11 H Lehmann and K Pohlmeyer, Comm Math Phys 20, 101 (1971)

12 C J Isham, A Salam and J Strathdee, Phys Rev D5, 2548 (1972)

13 Tulsi Dass and Radhey Shyam, 'Conformal Invariant Finite Quantum Electrodynamics', (to be published in Phys Rev)

14 A Salam and J Strathdee, Phys Rev 184 1760 (1969)

15 E Cunningham, Proc Lond Math Soc 8 77 (1910)

16 H Bateman, Proc Lond Math Soc 8, 22 (1910)

17 L Page, Phys Rev 49, 254 (1936)

18 P A M Dirac, Annals of Physics, 37, 429 (1936)

19 T Fulton, R Rohrlich, L Witten, Nuovo Cimento 26, 653 (1962)

20 L Castell, Nuovo Cimento 46A, 1 (1966)

21 H A Kastrup, Phys Rev 140, B183 (1965), Phys Rev 142, 1060 (1960), Phys Rev 150, 1183 (1966)

22 J ess, Nuovo Cimento 18, 1086 (1960)

23 Some of these are

A Kihlberg, V F Muller and F Halbwachs, Comm Math Phys 3, 194 (1966)

R Raczka, N Limic, J Neicerde, J Math Phys 7, 1861, 2026 (1966), 8, 1079 (1967)

A Steve and P G Sona, Nuovo Cimento 32, 473 (1964)
Y Murai, Prog Theo Phys 9, 147 (1953)

24 G Mack and A Salam, Ann Phys (N Y) 53, (1969)

25 A Salam and J Strathdee, Phys Rev 184, 1760 (1969)

26 C J Isham, A Salam and J Strathdee Phys Lett 31B, 300 (1970)

27 J Ellis, Nucl Phys B22, 478 (1970)

28 S Weinberg, 'Gravitation and Cosmology Principles and Applications of the General Theory of Relativity', John Wiley and Sons (1972)

29 J D Bjorken and S D Drell, 'Relativistic Quantum Mechanics', McGraw Hill (1964)

30 I M Gel'fand and G E Shilov, 'Generalized Functions', Vol I (Academic Press 1964)

31 Bateman's Table of Higher Transcendental Functions', Vol I, Ed by A Erdelyi et al, McGraw Hill (1953)

32 J M Jauch and F Rohrlich, 'Theory of Photons and Electrons', Addison Wesley (1955), C G Bolloni, J J Giambiagi and A Gonzales Dominques, Nuovo Cimento 31, 550 (1964)

33 J M Jauch and F Rohrlich, Ref 32

34 J Schwinger, Phys Rev 75, 651 (1948)

35 K Johnson, Lectures on Quantum Electrodynamics in Brandeis Summer School (1964), Prentice Hall (1965)

36 N N Bogoliubov and D V Shirkov, 'Introduction to the Theory of Quantized Fields', Interscience Publishers (1959)

A 53972

Date Slip A 53972

This book is to be returned on the date last stamped

CD 6 72 9